

POSITIVITY PROPERTIES FOR CANONICAL BASES OF MODIFIED QUANTUM AFFINE \mathfrak{sl}_n

QIANG FU[†] AND TOSHIKI SHOJI

ABSTRACT. The positivity property for canonical bases asserts that the structure constants of the multiplication for the canonical basis are in $\mathbb{N}[v, v^{-1}]$. Let \mathbf{U} be the quantum group over $\mathbb{Q}(v)$ associated with a symmetric Cartan datum. The positivity property for the positive part \mathbf{U}^+ of \mathbf{U} was proved by Lusztig. He conjectured that the positivity property holds for the modified form $\dot{\mathbf{U}}$ of \mathbf{U} . In this paper, we prove that the structure constants for the canonical basis of $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$ coincide with certain structure constants for the canonical basis of $\mathbf{U}(\widehat{\mathfrak{sl}}_N)^+$ for $n < N$. In particular, the positivity property for $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$ follows from the positivity property for $\mathbf{U}(\widehat{\mathfrak{sl}}_N)^+$.

1. INTRODUCTION

Let \mathbf{U} be the quantum group over $\mathbb{Q}(v)$ associated with a Cartan datum (I, \cdot) , where v is an indeterminate. It is known by Lusztig and Kashiwara that the positive part \mathbf{U}^+ of a quantum enveloping algebra \mathbf{U} has a canonical basis with remarkable properties (see Kashiwara [K], Lusztig [L1, L2, L5]). Among them, the deepest one should be the positivity property for the canonical basis of \mathbf{U}^+ proved by Lusztig [L1, L2], [L5, 14.4.13], which asserts that the structure constants of the multiplication for the canonical basis of \mathbf{U}^+ are in $\mathbb{N}[v, v^{-1}]$ in the case where the Cartan datum (I, \cdot) is symmetric.

Let $\dot{\mathbf{U}}$ be the modified form of \mathbf{U} . The algebra $\dot{\mathbf{U}}$ is an associative algebra without unity and the category of \mathbf{U} -modules of type 1 is equivalent to the category of unital $\dot{\mathbf{U}}$ -modules. The canonical basis $\dot{\mathbf{B}}$ of $\dot{\mathbf{U}}$ was constructed by Lusztig [L4, L5]. In [L4, Section 11] and [L5, 25.4.2], he conjectured that the structure constants of the multiplication for $\dot{\mathbf{B}}$ are in $\mathbb{N}[v, v^{-1}]$, i.e., the positivity property holds for $\dot{\mathbf{U}}$, in the case where the Cartan datum (I, \cdot) is symmetric.

Let $\mathcal{S}_\Delta(n, r)$ be the affine quantum Schur algebra over $\mathbb{Q}(v)$ (see [GV], [G2] and [L6]). An explicit algebra homomorphism ζ_r from $\mathbf{U}(\widehat{\mathfrak{sl}}_n)$ to $\mathcal{S}_\Delta(n, r)$ was constructed by Ginzburg–Vasserot [GV], Lusztig [L6]. According to [L6, 8.2] the map $\zeta_r : \mathbf{U}(\widehat{\mathfrak{sl}}_n) \rightarrow \mathcal{S}_\Delta(n, r)$ is not surjective in the case where $n \leq r$. In turn, it is proved by Deng–Du–Fu [DDF, 3.8.1] that the map ζ_r can be extended to a surjective algebra homomorphism from $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$ to $\mathcal{S}_\Delta(n, r)$, where $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$ is the quantum loop algebra of $\widehat{\mathfrak{gl}}_n$. On the other hand, the quantum Schur algebra $\mathcal{S}(n, r)$ is known to be a quotient of the quantum algebra $\mathbf{U}(\mathfrak{sl}_n)$. The canonical basis of $\mathcal{S}(n, r)$ was

[†] Supported by the National Natural Science Foundation of China, Fok Ying Tung Education Foundation.

defined by Beilinson–Lusztig–MacPherson [BLM] and the positivity property for the canonical basis of $\mathcal{S}(n, r)$ was proved by Green in [G1]. The canonical basis $\mathbf{B}(n, r)$ of the affine quantum Schur algebra $\mathcal{S}_\Delta(n, r)$ was defined in [L6]. Lusztig gave in [L6, 4.5] a sketch of the proof of the positivity property for $\mathbf{B}(n, r)$ based on the property of Kazhdan–Lusztig basis of affine Hecke algebras of type A .

In this paper, we show that there exist good relations among canonical bases of the three algebras $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$, $\mathcal{S}_\Delta(n, r)$ and $\mathbf{U}(\widehat{\mathfrak{sl}}_N)^+$. In Theorem 4.8 we prove that the structure constants for $\mathbf{B}(n, r)$ are determined by the structure constants for the canonical basis $\mathbf{B}(N)^{\text{ap}}$ of $\mathbf{U}(\widehat{\mathfrak{sl}}_N)^+$ for $n < N$. Then the positivity property for $\mathbf{B}(n, r)$ follows from the positivity property for $\mathbf{B}(N)^{\text{ap}}$. This gives an alternate approach for the positivity property of $\mathbf{B}(n, r)$. Using Theorem 4.8, we prove in Theorem 5.4 that the structure constants for the canonical basis $\dot{\mathbf{B}}(n)$ of $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$ are determined by the structure constants for the canonical basis $\mathbf{B}(N)^{\text{ap}}$ of $\mathbf{U}(\widehat{\mathfrak{sl}}_N)^+$ for $n < N$. Thus the positivity property for $\dot{\mathbf{B}}(n)$ follows from the positivity property for $\mathbf{B}(N)^{\text{ap}}$. We also discuss in Theorem 6.3 a certain weak positivity property for $\dot{\mathfrak{D}}_\Delta(n)$, where $\dot{\mathfrak{D}}_\Delta(n)$ is the modified quantum affine \mathfrak{gl}_n .

Notation: For a positive integer n , let $\Theta_\Delta(n)$ (resp., $\tilde{\Theta}_\Delta(n)$) be the set of all matrices $A = (a_{i,j})_{i,j \in \mathbb{Z}}$ with $a_{i,j} \in \mathbb{N}$ (resp. $a_{i,j} \in \mathbb{Z}$, $a_{i,j} \geq 0$ for all $i \neq j$) such that

- (a) $a_{i,j} = a_{i+n,j+n}$ for $i, j \in \mathbb{Z}$;
- (b) for every $i \in \mathbb{Z}$, both sets $\{j \in \mathbb{Z} \mid a_{i,j} \neq 0\}$ and $\{j \in \mathbb{Z} \mid a_{j,i} \neq 0\}$ are finite.

Let $\Theta_\Delta^+(n) = \{A \in \Theta_\Delta(n) \mid a_{i,j} = 0 \text{ for } i \geq j\}$. For $r \geq 0$, let $\Theta_\Delta(n, r) = \{A \in \Theta_\Delta(n) \mid \sigma(A) = r\}$, where $\sigma(A) = \sum_{1 \leq i \leq n, j \in \mathbb{Z}} a_{i,j}$. For $i, j \in \mathbb{Z}$ let $E_{i,j}^\Delta \in \Theta_\Delta(n)$ be the matrix $(e_{k,l}^{i,j})_{k,l \in \mathbb{Z}}$ defined by

$$e_{k,l}^{i,j} = \begin{cases} 1 & \text{if } k = i + sn, l = j + sn \text{ for some } s \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbb{Z}_\Delta^n = \{(\lambda_i)_{i \in \mathbb{Z}} \mid \lambda_i \in \mathbb{Z}, \lambda_i = \lambda_{i-n} \text{ for } i \in \mathbb{Z}\}$ and $\mathbb{N}_\Delta^n = \{(\lambda_i)_{i \in \mathbb{Z}} \in \mathbb{Z}_\Delta^n \mid \lambda_i \geq 0 \text{ for } i \in \mathbb{Z}\}$. \mathbb{Z}_Δ^n has a natural structure of abelian group. For $r \geq 0$ let $\Lambda_\Delta(n, r) = \{\lambda \in \mathbb{N}_\Delta^n \mid \sigma(\lambda) = r\}$, where $\sigma(\lambda) = \sum_{1 \leq i \leq n} \lambda_i$.

Let $\mathcal{Z} = \mathbb{Z}[v, v^{-1}]$, where v is an indeterminate.

2. PRELIMINARIES

2.1. Let $\Delta(n)$ ($n \geq 2$) be the cyclic quiver with vertex set $I = \mathbb{Z}/n\mathbb{Z}$ and arrow set $\{i \rightarrow i+1 \mid i \in I\}$. We identify I with $\{1, 2, \dots, n\}$. Let \mathbb{F} be a field. For $i \in I$ and $j \in \mathbb{Z}$ with $i < j$, let S_i denote the one-dimensional representation of $\Delta(n)$ with $(S_i)_i = \mathbb{F}$ and $(S_i)_k = 0$ for $i \neq k$ and $M^{i,j}$ the unique indecomposable nilpotent representation of length $j - i$ with top S_i .

For $A \in \Theta_\Delta^+(n)$ let $\mathbf{d}(A) \in \mathbb{N}I$ be the dimension vector of $M(A)$, where

$$M(A) = M_{\mathbb{F}}(A) = \bigoplus_{\substack{1 \leq i \leq n \\ i < j, j \in \mathbb{Z}}} a_{i,j} M^{i,j}.$$

We will identify naturally $\mathbb{N}I$ with \mathbb{N}_Δ^n . The Euler form associated with the cyclic quiver $\Delta(n)$ is the bilinear form $\langle -, - \rangle: \mathbb{Z}_\Delta^n \times \mathbb{Z}_\Delta^n \rightarrow \mathbb{Z}$ defined by $\langle \lambda, \mu \rangle = \sum_{1 \leq i \leq n} \lambda_i \mu_i - \sum_{1 \leq i \leq n} \lambda_i \mu_{i+1}$ for $\lambda, \mu \in \mathbb{Z}_\Delta^n$.

By Ringel [R], for $A, B, C \in \Theta_\Delta^+(n)$, there is a polynomial $\varphi_{A,B}^C \in \mathbb{Z}[v^2]$ such that, for any finite field \mathbb{F}_q , $\varphi_{A,B}^C|_{v^2=q}$ is equal to the number of submodules N of $M_{\mathbb{F}_q}(C)$ satisfying $N \cong M_{\mathbb{F}_q}(B)$ and $M_{\mathbb{F}_q}(C)/N \cong M_{\mathbb{F}_q}(A)$.

Let $\mathfrak{D}_\Delta(n)$ be the double Ringel–Hall algebra of the cyclic quiver $\Delta(n)$ introduced in [DDF, (2.1.3.2)] (see also [X]). It was proved in [DDF, 2.5.3] that $\mathfrak{D}_\Delta(n)$ is isomorphic to the quantum loop algebra $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$. According to [DDF, 2.6.1, 2.6.3(5) and 3.9.2] we have the following result.

Lemma 2.1. *The algebra $\mathfrak{D}_\Delta(n)$ is the algebra over $\mathbb{Q}(v)$ generated by u_A^+ , $K_i^{\pm 1}$, u_A^- ($A \in \Theta_\Delta^+(n)$, $i \in I$) subject to the following relations:*

- (1) $K_i K_j = K_j K_i$, $K_i K_i^{-1} = 1$, $u_0^+ = u_0^- = 1$;
- (2) $K^{\mathbf{j}} u_A^+ = v^{\langle \mathbf{d}(A), \mathbf{j} \rangle} u_A^+ K^{\mathbf{j}}$, $u_A^- K^{\mathbf{j}} = v^{\langle \mathbf{d}(A), \mathbf{j} \rangle} K^{\mathbf{j}} u_A^-$, where $K^{\mathbf{j}} = K_1^{j_1} \cdots K_n^{j_n}$ for $\mathbf{j} \in \mathbb{Z}_\Delta^n$;
- (3) $u_A^+ u_B^+ = \sum_{C \in \Theta_\Delta^+(n)} v^{\langle \mathbf{d}(A), \mathbf{d}(B) \rangle} \varphi_{A,B}^C u_C^+$;
- (4) $u_A^- u_B^- = \sum_{C \in \Theta_\Delta^+(n)} v^{\langle \mathbf{d}(B), \mathbf{d}(A) \rangle} \varphi_{B,A}^C u_C^-$;
- (5) commutator relations: for all $\lambda, \mu \in \mathbb{N}_\Delta^n$,

$$v^{\langle \mu, \mu \rangle} \sum_{\substack{\alpha, \beta \in \mathbb{N}_\Delta^n \\ \lambda - \alpha = \mu - \beta \geq 0}} \varphi_{\lambda, \mu}^{\alpha, \beta} v^{\langle \beta, \lambda + \mu - \beta \rangle} \tilde{K}^{\mu - \beta} u_{A_\beta}^- u_{A_\alpha}^+ = v^{\langle \mu, \lambda \rangle} \sum_{\substack{\alpha, \beta \in \mathbb{N}_\Delta^n \\ \lambda - \alpha = \mu - \beta \geq 0}} \varphi_{\lambda, \mu}^{\alpha, \beta} v^{\langle \mu - \beta, \alpha \rangle + \langle \mu, \beta \rangle} \tilde{K}^{\beta - \mu} u_{A_\alpha}^+ u_{A_\beta}^-,$$

where $\tilde{K}^\nu := (\tilde{K}_1)^{\nu_1} \cdots (\tilde{K}_n)^{\nu_n}$ with $\tilde{K}_i = K_i K_{i+1}^{-1}$ for $\nu \in \mathbb{Z}_\Delta^n$, and

$$\varphi_{\lambda, \mu}^{\alpha, \beta} = v^{2 \sum_{1 \leq i \leq n} (\lambda_i - \alpha_i)(1 - \alpha_i - \beta_i)} \prod_{\substack{1 \leq i \leq n \\ 0 \leq s \leq \lambda_i - \alpha_i - 1}} \frac{1}{v^{2(\lambda_i - \alpha_i) - v^{2s}}}.$$

Note that the set $\{u_A^+ K^{\mathbf{j}} u_B^- \mid A, B \in \Theta_\Delta^+(n), \mathbf{j} \in \mathbb{Z}_\Delta^n\}$ forms a $\mathbb{Q}(v)$ -basis of $\mathfrak{D}_\Delta(n)$.

2.2. We now recall the definition of affine quantum Schur algebras following [L6]. Let \mathbb{F} be a field and fix an $\mathbb{F}[\varepsilon, \varepsilon^{-1}]$ -free module V of rank $r \in \mathbb{N}$, where ε is an indeterminate. A lattice in V is, by definition, a free $\mathbb{F}[\varepsilon]$ -submodule L of V satisfying $V = L \otimes_{\mathbb{F}[\varepsilon]} \mathbb{F}[\varepsilon, \varepsilon^{-1}]$. Let $\mathcal{F}_\Delta = \mathcal{F}_{\Delta, n}$ be the set of all filtrations $\mathbf{L} = (L_i)_{i \in \mathbb{Z}}$ of lattices, where each L_i is a lattice in V such that $L_{i-1} \subseteq L_i$ and $L_{i-n} = \varepsilon L_i$, for all $i \in \mathbb{Z}$. The group G of automorphisms of the $\mathbb{F}[\varepsilon, \varepsilon^{-1}]$ -module V acts on \mathcal{F}_Δ by $g \cdot \mathbf{L} = (g(L_i))_{i \in \mathbb{Z}}$ for $g \in G$ and $\mathbf{L} \in \mathcal{F}_\Delta$. The group G acts on $\mathcal{F}_\Delta \times \mathcal{F}_\Delta$ by $g \cdot (\mathbf{L}, \mathbf{L}') = (g \cdot \mathbf{L}, g \cdot \mathbf{L}')$.

Recall the set $\Theta_\Delta(n, r)$ given in §1. By [L6, 1.5] there is a bijection between the set of G -orbits in $\mathcal{F}_\Delta \times \mathcal{F}_\Delta$ and $\Theta_\Delta(n, r)$ by sending $(\mathbf{L}, \mathbf{L}')$ to $A = (a_{i,j})_{i,j \in \mathbb{Z}}$, where $a_{i,j} = \dim_{\mathbb{F}} L_i \cap L'_j / (L_{i-1} \cap L'_j + L_i \cap L'_{j-1})$. Let $\mathcal{O}_A \subseteq \mathcal{F}_\Delta \times \mathcal{F}_\Delta$ be the G -orbit corresponding to the matrix $A \in \Theta_\Delta(n, r)$.

Let $\mathbb{F} = \mathbb{F}_q$ be the finite field of q elements. For $A, A', A'' \in \Theta_\Delta(n, r)$ and $(\mathbf{L}, \mathbf{L}'') \in \mathcal{O}_{A''}$ let $\nu_{A,A',A'';q} = \#\{\mathbf{L}' \in \mathcal{F}_\Delta \mid (\mathbf{L}, \mathbf{L}') \in \mathcal{O}_A, (\mathbf{L}', \mathbf{L}'') \in \mathcal{O}_{A''}\}$. By [L6, 1.8], there exists a polynomial $\nu_{A,A',A''} \in \mathbb{Z}$ in v^2 such that $\nu_{A,A',A''}|_{v^2=q} = \nu_{A,A',A'';q}$ for any q , a power of a prime number.

Let $\mathcal{S}_\Delta(n, r)_{\mathcal{Z}}$ be the free \mathcal{Z} -module with basis $\{e_A \mid A \in \Theta_\Delta(n, r)\}$. According to [L6, 1.9] there is a unique associative \mathcal{Z} -algebra structure on $\mathcal{S}_\Delta(n, r)_{\mathcal{Z}}$ with multiplication $e_A e_{A'} = \sum_{A'' \in \Theta_\Delta(n, r)} \nu_{A,A',A''} e_{A''}$. Let $\mathcal{S}_\Delta(n, r) = \mathcal{S}_\Delta(n, r)_{\mathcal{Z}} \otimes \mathbb{Q}(v)$. The algebras $\mathcal{S}_\Delta(n, r)_{\mathcal{Z}}$ and $\mathcal{S}_\Delta(n, r)$ are called affine quantum Schur algebras.

For $A \in \Theta_\Delta(n, r)$ let

$$(2.1) \quad [A] = v^{-d_A} e_A, \quad \text{where} \quad d_A = \sum_{1 \leq i \leq n, i \geq k, j < l} a_{i,j} a_{k,l}$$

According to [L6, 1.11], the \mathcal{Z} -linear map

$$(2.2) \quad \tau_r : \mathcal{S}_\Delta(n, r)_{\mathcal{Z}} \longrightarrow \mathcal{S}_\Delta(n, r)_{\mathcal{Z}}, \quad [A] \longmapsto [{}^t A]$$

is an algebra anti-involution, where ${}^t A$ is the transpose of A .

2.3. Let $\mathfrak{S}_{\Delta, r}$ be the group consisting of all permutations $w : \mathbb{Z} \rightarrow \mathbb{Z}$ such that $w(i+r) = w(i) + r$ for $i \in \mathbb{Z}$. The extended affine Hecke algebra $\mathcal{H}_\Delta(r)_{\mathcal{Z}}$ of affine type A over \mathcal{Z} is the (unital) \mathcal{Z} -algebra with basis $\{T_w\}_{w \in \mathfrak{S}_{\Delta, r}}$, and multiplication defined by

$$\begin{cases} T_{s_i}^2 = (v^2 - 1)T_{s_i} + v^2, & \text{for } 1 \leq i \leq r \\ T_w T_{w'} = T_{ww'}, & \text{if } \ell(ww') = \ell(w) + \ell(w'), \end{cases}$$

where $s_i \in \mathfrak{S}_{\Delta, r}$ is defined by setting $s_i(j) = j$ for $j \not\equiv i, i+1 \pmod{r}$, $s_i(j) = j-1$ for $j \equiv i+1 \pmod{r}$ and $s_i(j) = j+1$ for $j \equiv i \pmod{r}$, and $\ell(w)$ is the length of w .

Recall the set $\Lambda_\Delta(n, r)$ given in §1. Let \mathfrak{S}_r be the subgroup of $\mathfrak{S}_{\Delta, r}$ generated by s_i for $1 \leq i \leq r-1$, which is isomorphic to the symmetric group of degree r . For $\lambda \in \Lambda_\Delta(n, r)$, let $\mathfrak{S}_\lambda := \mathfrak{S}_{(\lambda_1, \dots, \lambda_n)}$ be the corresponding standard Young subgroup of \mathfrak{S}_r and let $x_\lambda = \sum_{w \in \mathfrak{S}_\lambda} T_w \in \mathcal{H}_\Delta(r)_{\mathcal{Z}}$. For $\lambda, \mu \in \Lambda_\Delta(n, r)$, let $\mathcal{D}_\lambda^\Delta = \{d \mid d \in \mathfrak{S}_{\Delta, r}, \ell(wd) = \ell(w) + \ell(d) \text{ for } w \in \mathfrak{S}_\lambda\}$ and $\mathcal{D}_{\lambda, \mu}^\Delta = \mathcal{D}_\lambda^\Delta \cap \mathcal{D}_\mu^{\Delta^{-1}}$. For $\lambda, \mu \in \Lambda_\Delta(n, r)$ and $d \in \mathcal{D}_{\lambda, \mu}^\Delta$ define $\phi_{\lambda, \mu}^d \in \text{End}_{\mathcal{H}_\Delta(r)_{\mathcal{Z}}}(\bigoplus_{\lambda \in \Lambda_\Delta(n, r)} x_\lambda \mathcal{H}_\Delta(r)_{\mathcal{Z}})$ by

$$\phi_{\lambda, \mu}^d(x_\nu h) = \delta_{\mu\nu} \sum_{w \in \mathfrak{S}_\lambda d \mathfrak{S}_\mu} T_w h$$

for $\nu \in \Lambda_\Delta(n, r)$ and $h \in \mathcal{H}_\Delta(r)_{\mathcal{Z}}$.

For $\lambda \in \Lambda_\Delta(n, r)$, $1 \leq i \leq n$ and $k \in \mathbb{Z}$ let

$$(2.3) \quad R_{i+kn}^\lambda = \{\lambda_{k,i-1} + 1, \lambda_{k,i-1} + 2, \dots, \lambda_{k,i-1} + \lambda_i = \lambda_{k,i}\},$$

where $\lambda_{k,i-1} = kr + \sum_{1 \leq t \leq i-1} \lambda_t$. By Varagnolo–Vasserot [VV, 7.4] (see also [DF1, 9.2]), there is a bijective map

$$(2.4) \quad \jmath_\Delta : \{(\lambda, d, \mu) \mid d \in \mathcal{D}_{\lambda, \mu}^\Delta, \lambda, \mu \in \Lambda_\Delta(n, r)\} \longrightarrow \Theta_\Delta(n, r)$$

sending (λ, d, μ) to the matrix $A = (|R_k^\lambda \cap dR_l^\mu|)_{k,l \in \mathbb{Z}}$. Varagnolo–Vasserot showed in [VV] that there is an algebra isomorphism

$$\mathfrak{h} : \text{End}_{\mathcal{H}_\Delta(r)_\mathbb{Z}} \left(\bigoplus_{\lambda \in \Lambda_\Delta(n, r)} x_\lambda \mathcal{H}_\Delta(r)_\mathbb{Z} \right) \rightarrow \mathcal{S}_\Delta(n, r)_\mathbb{Z}$$

such that $\mathfrak{h}(\phi_{\lambda, \mu}^d) = e_A$, where $A = \jmath_\Delta(\lambda, d, \mu)$. We identify $\text{End}_{\mathcal{H}_\Delta(r)_\mathbb{Z}} (\bigoplus_{\lambda \in \Lambda_\Delta(n, r)} x_\lambda \mathcal{H}_\Delta(r)_\mathbb{Z})$ with $\mathcal{S}_\Delta(n, r)_\mathbb{Z}$ via \mathfrak{h} .

2.4. It was shown in [DDF] that the double Ringel–Hall algebra $\mathfrak{D}_\Delta(n)$ and the affine quantum Schur algebra $\mathcal{S}_\Delta(n, r)$ are related by a surjective algebra homomorphism ζ_r . Let $\Theta_\Delta^\pm(n) := \{A \in \Theta_\Delta(n) \mid a_{i,j} = 0 \text{ for } i = j\}$. For $A \in \Theta_\Delta^\pm(n)$ and $\mathbf{j} \in \mathbb{Z}_\Delta^n$, define $A(\mathbf{j}, r) \in \mathcal{S}_\Delta(n, r)$ by

$$A(\mathbf{j}, r) = \begin{cases} \sum_{\lambda \in \Lambda_\Delta(n, r - \sigma(A))} v^{\lambda \cdot \mathbf{j}} [A + \text{diag}(\lambda)], & \text{if } \sigma(A) \leq r; \\ 0, & \text{otherwise,} \end{cases}$$

where $\lambda \cdot \mathbf{j} = \sum_{1 \leq i \leq n} \lambda_i j_i$. For $A \in \Theta_\Delta^+(n)$ let

$$\tilde{u}_A^\pm = v^{\dim \text{End}(M(A)) - \dim M(A)} u_A^\pm.$$

We have the following result.

Theorem 2.2 ([DDF, 3.6.3, 3.8.1]). *For $r \geq 0$, the linear map $\zeta_r : \mathfrak{D}_\Delta(n) \rightarrow \mathcal{S}_\Delta(n, r)$ satisfying*

$$\zeta_r(K^{\mathbf{j}}) = 0(\mathbf{j}, r), \quad \zeta_r(\tilde{u}_A^+) = A(\mathbf{0}, r), \quad \text{and} \quad \zeta_r(\tilde{u}_A^-) = ({}^t A)(\mathbf{0}, r),$$

for all $\mathbf{j} \in \mathbb{Z}_\Delta^n$ and $A \in \Theta_\Delta^+(n)$, is a surjective algebra homomorphism.

3. CANONICAL BASES FOR AFFINE QUANTUM SCHUR ALGEBRAS

3.1. Let W_r be the subgroup of $\mathfrak{S}_{\Delta, r}$ generated by s_i for $1 \leq i \leq r$. For $i, j \in \mathbb{Z}$ such that $i \not\equiv j \pmod{r}$, define $(i, j) \in \mathfrak{S}_{\Delta, r}$ by setting $(i, j)(k) = k$ for $k \not\equiv i, j \pmod{r}$, $(i, j)(k) = j + k - i$ for $k \equiv i \pmod{r}$ and $(i, j)(k) = i + k - j$ for $k \equiv j \pmod{r}$. Note that $(i, j) \in W_r$ for all i, j . By definition we have $(i, j) = (i + tr, j + tr)$ for $t \in \mathbb{Z}$ and $(i, i + 1) = s_i$. Let

$$T = \bigcup_{w \in W_r, 1 \leq i \leq r} w s_i w^{-1} = \{(i, j) \in W_r \mid 1 \leq i \leq r, i, j \in \mathbb{Z}, i < j, i \not\equiv j \pmod{r}\}.$$

For $y, w \in W_r$, we write $y \leq w$ if there exist $t_i \in T$ ($1 \leq i \leq m$) for some $m \in \mathbb{N}$ such that $w = t_1 t_2 \cdots t_m y$ and $\ell(t_i t_{i+1} \cdots t_m y) > \ell(t_{i+1} t_{i+2} \cdots t_m y)$ for $1 \leq i \leq m$. The partial ordering \leq on W_r is called the Bruhat order. Let ρ be the permutation of \mathbb{Z} sending j to $j+1$ for all $j \in \mathbb{Z}$. Then we have $\mathfrak{S}_{\Delta, r} = \langle \rho \rangle \ltimes W_r$, where $\langle \rho \rangle \cong \mathbb{Z}$ is the subgroup of $\mathfrak{S}_{\Delta, r}$ generated by ρ . The Bruhat order on W_r can be extended to $\mathfrak{S}_{\Delta, r}$ by define $\rho^i y \leq \rho^j w$ (for $y, w \in W_r$) if and only if $i = j$ and $y \leq w$.

Let $\bar{\cdot} : \mathcal{H}_{\Delta}(r)_{\mathcal{Z}} \rightarrow \mathcal{H}_{\Delta}(r)_{\mathcal{Z}}$ be the ring involution defined by $\bar{v} = v^{-1}$ and $\bar{T}_w = T_{w^{-1}}$. Let $\mathcal{H}(W_r)$ be the \mathcal{Z} -subalgebra of $\mathcal{H}_{\Delta}(r)_{\mathcal{Z}}$ generated by T_{s_i} for $1 \leq i \leq r$. Let $\{C'_w \mid w \in W_r\}$ be the Kazhdan–Lusztig basis of $\mathcal{H}(W_r)$ defined in [KL, 1.1(c)]. For $y, w \in W_r$ and $a, b \in \mathbb{Z}$ let $P_{\rho^a y, \rho^b w} = \delta_{a,b} P_{y,w}$, where $P_{y,w} \in \mathcal{Z}$ is the Kazhdan–Lusztig polynomial. For $w = \rho^a x \in \mathfrak{S}_{\Delta, r}$ with $a \in \mathbb{Z}$ and $x \in W_r$, let $C'_w = T_{\rho^a} C'_x$. Then for $w \in \mathfrak{S}_{\Delta, r}$ we have $\overline{C'_w} = C'_w$ and

$$C'_w = \sum_{y \leq w, y \in \mathfrak{S}_{\Delta, r}} v^{\ell(y) - \ell(w)} P_{y,w} \tilde{T}_y$$

where $\tilde{T}_y = v^{-\ell(y)} T_y$. The set $\{C'_w \mid w \in \mathfrak{S}_{\Delta, r}\}$ is called the canonical basis of $\mathcal{H}_{\Delta}(r)_{\mathcal{Z}}$.

For $d \in \mathcal{D}_{\lambda, \mu}^{\Delta}$ let $T_{\mathfrak{S}_{\lambda} d \mathfrak{S}_{\mu}} = \sum_{w \in \mathfrak{S}_{\lambda} d \mathfrak{S}_{\mu}} T_w$ and $\tilde{T}_{\mathfrak{S}_{\lambda} d \mathfrak{S}_{\mu}} = v^{-\ell(d^+)} T_{\mathfrak{S}_{\lambda} d \mathfrak{S}_{\mu}}$, where d^+ is the unique longest element in $\mathfrak{S}_{\lambda} d \mathfrak{S}_{\mu}$. According to [C, (1.10)] and [DDPW, 4.35] we have the following result.

Lemma 3.1. *For $\lambda, \mu \in \Lambda_{\Delta}(n, r)$ and $d \in \mathcal{D}_{\lambda, \mu}^{\Delta}$ we have*

$$C'_{d^+} = \sum_{\substack{y \in \mathcal{D}_{\lambda, \mu}^{\Delta} \\ y \leq d}} v^{\ell(y^+) - \ell(d^+)} P_{y^+, d^+} \tilde{T}_{\mathfrak{S}_{\lambda} y \mathfrak{S}_{\mu}},$$

where y^+ is the unique longest element in $\mathfrak{S}_{\lambda} y \mathfrak{S}_{\mu}$.

3.2. We now recall the definition of canonical bases of affine quantum Schur algebras. Note that $C'_{w_{0, \lambda}} = v^{-\ell(w_{0, \lambda})} x_{\lambda}$, where $w_{0, \lambda}$ is the longest element in \mathfrak{S}_{λ} . We define a map $\bar{\cdot} : \mathcal{S}_{\Delta}(n, r)_{\mathcal{Z}} \rightarrow \mathcal{S}_{\Delta}(n, r)_{\mathcal{Z}}$ by $v \mapsto \bar{v} = v^{-1}$, $f \mapsto \bar{f}$, where for $f \in \text{Hom}_{\mathcal{H}_{\Delta}(r)_{\mathcal{Z}}}(x_{\mu} \mathcal{H}_{\Delta}(r)_{\mathcal{Z}}, x_{\lambda} \mathcal{H}_{\Delta}(r)_{\mathcal{Z}})$, $\bar{f} \in \text{Hom}_{\mathcal{H}_{\Delta}(r)_{\mathcal{Z}}}(x_{\mu} \mathcal{H}_{\Delta}(r)_{\mathcal{Z}}, x_{\lambda} \mathcal{H}_{\Delta}(r)_{\mathcal{Z}})$ is defined by $\bar{f}(C'_{w_{0, \mu}} h) = \overline{f(C'_{w_{0, \mu}})} h$ for $h \in \mathcal{H}_{\Delta}(r)_{\mathcal{Z}}$. Then the map $\bar{\cdot} : \mathcal{S}_{\Delta}(n, r)_{\mathcal{Z}} \rightarrow \mathcal{S}_{\Delta}(n, r)_{\mathcal{Z}}$ is a ring involution (cf. [D]).

For $A \in \Theta_{\Delta}(n)$ let $\text{ro}(A) = (\sum_{j \in \mathbb{Z}} a_{i,j})_{i \in \mathbb{Z}}$ and $\text{co}(A) = (\sum_{i \in \mathbb{Z}} a_{i,j})_{j \in \mathbb{Z}}$. For $A \in \tilde{\Theta}_{\Delta}(n)$ and $i \neq j \in \mathbb{Z}$, let

$$\sigma_{i,j}(A) = \begin{cases} \sum_{s \leq i, t \geq j} a_{s,t}, & \text{if } i < j; \\ \sum_{s \geq i, t \leq j} a_{s,t}, & \text{if } i > j. \end{cases}$$

For $A, B \in \tilde{\Theta}_{\Delta}(n)$, define $B \preccurlyeq A$ by the condition $\sigma_{i,j}(B) \leq \sigma_{i,j}(A)$ for all $i \neq j$. Put $B \prec A$ if $B \preccurlyeq A$ and, for some pair (i, j) with $i \neq j$, $\sigma_{i,j}(B) < \sigma_{i,j}(A)$. For $A, B \in \tilde{\Theta}_{\Delta}(n)$ define $B \sqsubseteq A$ if and only if $B \preccurlyeq A$, $\text{co}(B) = \text{co}(A)$ and $\text{ro}(B) = \text{ro}(A)$. Put $B \sqsubset A$ if $B \sqsubseteq A$ and $B \neq A$. According to [DF1, 6.1] we know that the order relation \sqsubseteq is a partial order relation on $\tilde{\Theta}_{\Delta}(n)$.

Lusztig proved in [L6] that there is a unique \mathcal{Z} -basis

$$(3.1) \quad \mathbf{B}(n, r) := \{\theta_{A,r} \mid A \in \Theta_\Delta(n, r)\}$$

for $\mathcal{S}_\Delta(n, r)_\mathcal{Z}$ such that $\overline{\theta_{A,r}} = \theta_{A,r}$ and

$$(3.2) \quad \theta_{A,r} - [A] \in \sum_{\substack{B \in \Theta_\Delta(n, r) \\ B \subset A}} v^{-1} \mathbb{Z}[v^{-1}][B],$$

(see also [DF3, 7.6]). The set $\mathbf{B}(n, r)$ is called the canonical basis of $\mathcal{S}_\Delta(n, r)_\mathcal{Z}$.

3.3. For $w \in \mathfrak{S}_{\Delta,r}$ let $\mathcal{L}(w) = \{(i, j) \in \mathbb{Z}^2 \mid 1 \leq i \leq r, i < j, w(i) > w(j)\}$ and $\mathcal{R}(w) = \{(i, j) \in \mathbb{Z}^2 \mid 1 \leq j \leq r, i < j, w(i) > w(j)\}$. The following result is given in [DDF, (3.2.1.1)] (see also [DF2, 5.2]).

Lemma 3.2. *For $w \in \mathfrak{S}_{\Delta,r}$, we have $\ell(w) = |\mathcal{L}(w)| = |\mathcal{R}(w)|$.*

For $i \in \mathbb{Z}$ the image of i in $\mathbb{Z}/r\mathbb{Z}$ will be denoted by \bar{i} . The following corollary can be proved by a standard argument by using Lemma 3.2. So we omit the proof.

Corollary 3.3. *Let $x \in \mathfrak{S}_{\Delta,r}$ and $i_0, j_0 \in \mathbb{Z}$ such that $i_0 < j_0$ and $\bar{i}_0 \neq \bar{j}_0$. Then we have $x < (i_0, j_0)x$ if and only if $x^{-1}(i_0) < x^{-1}(j_0)$, i.e. i_0 occurs in the left of j_0 in the sequence $(x(s))_{s \in \mathbb{Z}}$.*

For $i \in \mathbb{Z}$ let $(-\infty, i] = \{\mathbf{a} = (a_s)_{s \leq i} \mid a_s \in \mathbb{Z}\}$ and $[i, +\infty) = \{\mathbf{a} = (a_s)_{s \geq i} \mid a_s \in \mathbb{Z}\}$. If either $\mathbf{a}, \mathbf{b} \in (-\infty, i]$ or $\mathbf{a}, \mathbf{b} \in [i, +\infty)$, we write $\mathbf{a} \leq \mathbf{b}$ if $a_s \leq b_s$ for all s . Given $\mathbf{a} = (a_s) \in \mathbb{Z}_\Delta^n$ and an integer i we let $(a_s)_{s \leq i}^{\text{sorted}} = (b_s)_{s \leq i}$ such that $\{a_s \mid s \leq i\} = \{b_s \mid s \leq i\}$ and $b_{s-1} \leq b_s$ for $s \leq i$. Similarly we may define $(a_s)_{s \geq i}^{\text{sorted}}$ for $\mathbf{a} \in \mathbb{Z}_\Delta^n$ and $i \in \mathbb{Z}$.

By Corollary 3.3 we have the following result.

Corollary 3.4. *Let $y, w \in \mathfrak{S}_{\Delta,r}$. If $y \leq w$ then for any $i \in \mathbb{Z}$ we have $(y(s))_{s \leq i}^{\text{sorted}} \leq (w(s))_{s \leq i}^{\text{sorted}}$ and $(y(s))_{s \geq i}^{\text{sorted}} \geq (w(s))_{s \geq i}^{\text{sorted}}$.*

3.4. Recall the map \mathcal{J}_Δ defined in (2.4). Given $A \in \Theta_\Delta(n, r)$, write $y_A = w$ if $A = \mathcal{J}_\Delta(\lambda, w, \mu)$. For $A, B \in \Theta_\Delta(n, r)$, define $B \leq^{Bo} A$ by the condition $\text{ro}(B) = \text{ro}(A)$, $\text{co}(B) = \text{co}(A)$ and $y_B \leq y_A$. Put $B <^{Bo} A$ if $B \leq^{Bo} A$ and $B \neq A$. Then \leq^{Bo} is a partial order relation on $\Theta_\Delta(n, r)$.

Recall that V is a $\mathbb{F}[\varepsilon, \varepsilon^{-1}]$ -free module of rank $r \in \mathbb{N}$. Let $\{v_1, v_2, \dots, v_r\}$ be a fixed $\mathbb{F}[\varepsilon, \varepsilon^{-1}]$ -basis of V . We set $v_{i+kr} = \varepsilon^{-k} v_i$ for $1 \leq i \leq r$ and $k \in \mathbb{Z}$.

Lemma 3.5. *Let $A \in \Theta_\Delta(n, r)$, $\lambda = \text{ro}(A)$ and $\mu = \text{co}(A)$. Let $\mathbf{L}(A) = (L_i)_{i \in \mathbb{Z}}$ and $\mathbf{L}'(A) = (L'_i)_{i \in \mathbb{Z}}$ where*

$$L_{i+kn} = \text{span}_{\mathbb{F}} \left\{ v_a | a \in \bigcup_{t \leq i+kn} R_t^\lambda \right\} = \text{span}_{\mathbb{F}} \left\{ v_a | a \leq \sum_{1 \leq j \leq i} \lambda_j + kr \right\}$$

$$L'_{i+kn} = \text{span}_{\mathbb{F}} \left\{ v_{y_A(a)} | a \in \bigcup_{t \leq i+kn} R_t^\mu \right\} = \text{span}_{\mathbb{F}} \left\{ v_{y_A(a)} | a \leq \sum_{1 \leq j \leq i} \mu_j + kr \right\}$$

for $1 \leq i \leq n$ and $k \in \mathbb{Z}$. Then we have $(\mathbf{L}(A), \mathbf{L}'(A)) \in \mathcal{O}_A$.

Proof. By definition we have $L_i \cap L'_j = \text{span}_{\mathbb{F}} \{v_a | a \in \bigcup_{t \leq i} R_t^\lambda, a \in \bigcup_{t \leq j} y_A(R_t^\mu)\}$ for $i, j \in \mathbb{Z}$. Hence for $i, j \in \mathbb{Z}$ we have

$$\frac{L_i \cap L'_j}{L_{i-1} \cap L'_j + L_i \cap L'_{j-1}} = \text{span}_{\mathbb{F}} \{\bar{v}_a | a \in R_i^\lambda \cap y_A(R_j^\mu)\}.$$

The assertion follows. \square

Lemma 3.6. (1) *If $A, B \in \Theta_\Delta(n, r)$ and $B \leq^{Bo} A$ then $B \sqsubseteq A$.*

(2) *If $A, B \in \Theta_\Delta(n, r)$ and $B <^{Bo} A$ then $B \sqsubset A$.*

Proof. If $B \leq^{Bo} A$ then $\text{ro}(B) = \text{ro}(A)$, $\text{co}(B) = \text{co}(A)$ and $y_B \leq y_A$. We denote $\lambda = \text{ro}(B)$ and $\mu = \text{co}(B)$. Let $\mathbf{L} = \mathbf{L}(A) = \mathbf{L}(B)$, $\mathbf{L}' = \mathbf{L}'(A)$ and $\mathbf{L}'' = \mathbf{L}'(B)$. Then by Lemma 3.5 we have $(\mathbf{L}, \mathbf{L}') \in \mathcal{O}_A$ and $(\mathbf{L}', \mathbf{L}'') \in \mathcal{O}_B$. By definition for $i, j \in \mathbb{Z}$ we have

$$L_i / (L_i \cap L'_{j-1}) = \text{span}_{\mathbb{F}} \{\bar{v}_{y_A(a)} | y_A(a) \in \bigcup_{t \leq i} R_t^\lambda, a \in \bigcup_{t \geq j} R_t^\mu\},$$

$$L_i / (L_i \cap L''_{j-1}) = \text{span}_{\mathbb{F}} \{\bar{v}_{y_B(a)} | y_B(a) \in \bigcup_{t \leq i} R_t^\lambda, a \in \bigcup_{t \geq j} R_t^\mu\}$$

$$L'_j / (L_{i-1} \cap L'_j) = \text{span}_{\mathbb{F}} \{\bar{v}_{y_A(a)} | a \in \bigcup_{t \leq j} R_t^\mu, y_A(a) \in \bigcup_{t \geq i} R_t^\lambda\},$$

$$L''_j / (L_{i-1} \cap L''_j) = \text{span}_{\mathbb{F}} \{\bar{v}_{y_B(a)} | a \in \bigcup_{t \leq j} R_t^\mu, y_B(a) \in \bigcup_{t \geq i} R_t^\lambda\}.$$

Since $y_B \leq y_A$, by Corollary 3.4 we have $\dim_{\mathbb{F}}(L_i / (L_i \cap L''_{j-1})) \leq \dim_{\mathbb{F}}(L_i / (L_i \cap L'_{j-1}))$ and $\dim_{\mathbb{F}}(L''_j / (L_{i-1} \cap L''_j)) \leq \dim_{\mathbb{F}}(L'_j / (L_{i-1} \cap L'_j))$. Hence by [L6, 1.6(a)] we conclude that $B \sqsubseteq A$. Thus (1) holds. Now we assume $B <^{Bo} A$. Suppose that $B \not\sqsubset A$. Then by (1) we have $B \preceq A$ and $\text{ro}(B) = \text{ro}(A)$. Hence by [DF1, 6.1] we see that B and A have the same off diagonal entries. Since $\text{ro}(B) = \text{ro}(A)$ we must have $A = B$. This is a contradiction. Hence $B \prec A$. The assertion (2) follows. \square

3.5. For $A \in \Theta_\Delta(n, r)$ let y_A^+ be the unique longest element in $\mathfrak{S}_\lambda y_A \mathfrak{S}_\mu$, where $\lambda = \text{ro}(A)$ and $\mu = \text{co}(A)$. The following result is given in [DF3, 7.1].

Lemma 3.7. *For $A \in \Theta_\Delta(n, r)$ we have $\ell(y_A^+) = d_A + \ell(w_{0,\mu})$ where $\mu = \text{co}(A)$ and d_A is given in (2.1).*

For $\lambda, \mu \in \Lambda_\Delta(n, r)$ and $d \in \mathcal{D}_{\lambda,\mu}^\Delta$, define $\theta_{\lambda,\mu}^d \in \mathcal{S}_\Delta(n, r)_\mathcal{Z}$ as follows:

$$\theta_{\lambda,\mu}^d(x_\nu h) = \delta_{\mu\nu} v^{\ell(w_{0,\mu})} C'_{d^+} h,$$

where $\nu \in \Lambda_\Delta(n, r)$, $h \in \mathcal{H}_\Delta(r)_\mathcal{Z}$ and d^+ is the unique longest element in $\mathfrak{S}_\lambda d \mathfrak{S}_\mu$.

Proposition 3.8. *Assume $\lambda, \mu \in \Lambda_\Delta(n, r)$, $d \in \mathcal{D}_{\lambda,\mu}$ and $A = j_\Delta(\lambda, d, \mu) \in \Theta_\Delta(n, r)$. Then we have*

$$\theta_{\lambda,\mu}^d = \theta_{A,r} = \sum_{\substack{B \in \Theta_\Delta(n, r) \\ B \leq \widehat{B} \circ A}} v^{\ell(y_B^+) - \ell(y_A^+)} P_{y_B^+, y_A^+}[B],$$

where $P_{y_B^+, y_A^+}$ is the Kazhdan–Lusztig polynomial.

Proof. By definition we have $\overline{\theta_{\lambda,\mu}^d} = \theta_{\lambda,\mu}^d$. Furthermore, by Lemma 3.1 we conclude that

$$(3.3) \quad \theta_{\lambda,\mu}^d = \sum_{\substack{x \in \mathcal{D}_{\lambda,\mu}^\Delta \\ x \leq d}} v^{\ell(x^+) - \ell(d^+)} P_{x^+, d^+} \widetilde{\phi_{\lambda,\mu}^x},$$

where $\widetilde{\phi_{\lambda,\mu}^x} = v^{\ell(w_{0,\mu}) - \ell(x^+)} \phi_{\lambda,\mu}^x$. In addition, by Lemma 3.7 we have $[B] = \widetilde{\phi_{\lambda,\mu}^x}$ for $B \in \Theta_\Delta(n, r)$ with $B = j_\Delta(\lambda, x, \mu)$. Consequently, by Lemma 3.6 and the uniqueness of $\theta_{A,r}$ we conclude that $\theta_{A,r} = \theta_{\lambda,\mu}^d$. The assertion follows. \square

4. CONNECTION BETWEEN $\mathbf{B}(n, r)$ AND $\mathbf{B}(N)^{\text{ap}}$

4.1. Let $\mathbf{U}(\widehat{\mathfrak{sl}}_n)$ be the $\mathbb{Q}(v)$ -subalgebra of $\mathfrak{D}_\Delta(n)$ generated by the elements $u_{E_{i,i+1}^\Delta}^+$, $u_{E_{i+1,i}^\Delta}^-$ and $\widetilde{K}_i^{\pm 1}$ for $i \in I$. Then $\mathbf{U}(\widehat{\mathfrak{sl}}_n)$ is isomorphic to quantum affine $\widehat{\mathfrak{sl}}_n$. Let $\mathbf{U}(\widehat{\mathfrak{sl}}_n)^+$ be the $\mathbb{Q}(v)$ -subalgebra of $\mathbf{U}(\widehat{\mathfrak{sl}}_n)$ generated by the elements $u_{E_{i,i+1}^\Delta}^+$ for $i \in I$. Let $U(\widehat{\mathfrak{sl}}_n)_\mathcal{Z}^+$ be the \mathcal{Z} -subalgebra of $\mathbf{U}(\widehat{\mathfrak{sl}}_n)^+$ generated by $\widetilde{u}_{mE_{i,i+1}^\Delta}^+$ for $i \in I$ and $m \in \mathbb{N}$. The algebra $U(\widehat{\mathfrak{sl}}_n)_\mathcal{Z}^+$ is the \mathcal{Z} -form of $\mathbf{U}(\widehat{\mathfrak{sl}}_n)^+$.

Let $\mathfrak{D}_\Delta(n)_\mathcal{Z}^+ = \text{span}_\mathcal{Z}\{\widetilde{u}_A^+ \mid A \in \Theta_\Delta^+(n)\}$. Then $\mathfrak{D}_\Delta(n)_\mathcal{Z}^+$ is a \mathcal{Z} -subalgebra of $\mathfrak{D}_\Delta(n)$ and $U(\widehat{\mathfrak{sl}}_n)_\mathcal{Z}^+$ is a proper subalgebra of $\mathfrak{D}_\Delta(n)_\mathcal{Z}^+$. According to [VV, Prop 7.5], there is a unique \mathcal{Z} -basis

$$(4.1) \quad \mathbf{B}(n) := \{\theta_A^+ \mid A \in \Theta_\Delta^+(n)\}$$

for $\mathfrak{D}_\Delta(n)_\mathcal{Z}^+$ such that $\overline{\theta_A^+} = \theta_A^+$ and

$$(4.2) \quad \theta_A^+ - \widetilde{u}_A^+ \in \sum_{\substack{B \prec A, B \in \Theta_\Delta^+(n) \\ \mathbf{d}(B) = \mathbf{d}(A)}} v^{-1} \mathbb{Z}[v^{-1}] \widetilde{u}_B^+.$$

The set $\mathbf{B}(n)$ is called the canonical basis of $\mathfrak{D}_\Delta(n)_{\mathcal{Z}}^+$. For $A, B \in \Theta_\Delta^+(n)$ we write

$$(4.3) \quad \theta_A^+ \theta_B^+ = \sum_{C \in \Theta_\Delta^+(n)} f_{A,B,C} \theta_C^+,$$

where $f_{A,B,C} \in \mathcal{Z}$. Note that if $f_{A,B,C} \neq 0$ then $\mathbf{d}(C) = \mathbf{d}(A) + \mathbf{d}(B)$.

A matrix $A = (a_{i,j}) \in \Theta_\Delta(n)$ is said to be aperiodic if for every integer $l \neq 0$ there exists $1 \leq i \leq n$ such that $a_{i,i+l} = 0$. Let $\Theta_\Delta(n)^{\text{ap}}$ be the set of all aperiodic matrices in $\Theta_\Delta(n)$. Let $\Theta_\Delta^+(n)^{\text{ap}} = \Theta_\Delta^+(n) \cap \Theta_\Delta(n)^{\text{ap}}$.

By Lusztig [L3] we know that the set

$$(4.4) \quad \mathbf{B}(n)^{\text{ap}} := \{\theta_A^+ \mid A \in \Theta_\Delta^+(n)^{\text{ap}}\}$$

forms a \mathcal{Z} -basis for $U(\widehat{\mathfrak{sl}}_n)_{\mathcal{Z}}^+$ and is called the canonical basis of $U(\widehat{\mathfrak{sl}}_n)_{\mathcal{Z}}^+$. The following positivity result for $U(\widehat{\mathfrak{sl}}_n)_{\mathcal{Z}}^+$ was proved by Lusztig.

Theorem 4.1 ([L5, 14.4.13]). *For $A, B, C \in \Theta_\Delta^+(n)^{\text{ap}}$ we have $f_{A,B,C} \in \mathbb{N}[v, v^{-1}]$.*

4.2. Let $\mathfrak{D}_\Delta(n)_{\mathcal{Z}}^0$ be the \mathcal{Z} -subalgebra of $\mathfrak{D}_\Delta(n)$ generated by $K_i^{\pm 1}$ and $\begin{bmatrix} K_i & 0 \\ t & \end{bmatrix}$ for $1 \leq i \leq n$ and $t > 0$, where $\begin{bmatrix} K_i & 0 \\ t & \end{bmatrix} = \prod_{s=1}^t \frac{K_i v^{-s+1} - K_i^{-1} v^{s-1}}{v^s - v^{-s}}$. Let $\mathfrak{D}_\Delta(n)_{\mathcal{Z}}^{\geq 0} = \mathfrak{D}_\Delta(n)_{\mathcal{Z}}^+ \mathfrak{D}_\Delta(n)_{\mathcal{Z}}^0$. Then $\mathfrak{D}_\Delta(n)_{\mathcal{Z}}^{\geq 0}$ is a \mathcal{Z} -subalgebra of $\mathfrak{D}_\Delta(n)_{\mathcal{Z}}$.

Recall the map ζ_r defined in Theorem 2.2. Let $\mathcal{S}_\Delta(n, r)_{\mathcal{Z}}^{\geq 0}$ be the \mathcal{Z} -submodule of $\mathcal{S}_\Delta(n, r)_{\mathcal{Z}}$ spanned by the elements $A(\mathbf{0}, r)[\text{diag}(\lambda)]$ for $A \in \Theta_\Delta^+(n)$ and $\lambda \in \Lambda_\Delta(n, r)$. Since $\mathcal{S}_\Delta(n, r)_{\mathcal{Z}}^{\geq 0} = \zeta_r(\mathfrak{D}_\Delta(n)_{\mathcal{Z}}^{\geq 0})$, we conclude that $\mathcal{S}_\Delta(n, r)_{\mathcal{Z}}^{\geq 0}$ is a \mathcal{Z} -subalgebra of $\mathcal{S}_\Delta(n, r)_{\mathcal{Z}}$. The algebra $\mathcal{S}_\Delta(n, r)_{\mathcal{Z}}^{\geq 0}$ is called a Borel subalgebra of $\mathcal{S}_\Delta(n, r)_{\mathcal{Z}}$.

Lemma 4.2. *The set $\{\theta_{A+\text{diag}(\lambda), r} \mid A \in \Theta_\Delta^+(n), \lambda \in \Lambda_\Delta(n, r - \sigma(A))\}$ forms a \mathcal{Z} -basis of $\mathcal{S}_\Delta(n, r)_{\mathcal{Z}}^{\geq 0}$.*

Proof. By definition the set $\{[A + \text{diag}(\lambda)] \mid A \in \Theta_\Delta^+(n), \lambda \in \Lambda_\Delta(n, r - \sigma(A))\}$ forms a \mathcal{Z} -basis of $\mathcal{S}_\Delta(n, r)_{\mathcal{Z}}^{\geq 0}$. Furthermore, by (3.2), for $A \in \Theta_\Delta^+(n)$ and $\lambda \in \Lambda_\Delta(n, r - \sigma(A))$, we have

$$\theta_{A+\text{diag}(\lambda), r} - [A + \text{diag}(\lambda)] \in \sum_{\substack{B \in \Theta_\Delta^+(n), \mu \in \Lambda_\Delta(n, r - \sigma(B)) \\ B + \text{diag}(\mu) \sqsubset A + \text{diag}(\lambda)}} \mathcal{Z}[B + \text{diag}(\mu)].$$

The assertion follows. \square

According to [DF3, 7.7(2) and 7.9] we have the following result (see also [F2, 3.7]).

Lemma 4.3. *For $A \in \Theta_\Delta^+(n)$ we have $\zeta_r(\theta_A^+) = \sum_{\mu \in \Lambda_\Delta(n, r - \sigma(A))} \theta_{A+\text{diag}(\mu), r}$. In particular we have*

$$[\text{diag}(\lambda)]\zeta_r(\theta_A^+) = \begin{cases} \theta_{A+\text{diag}(\lambda - \text{ro}(A)), r} & \text{if } \lambda - \text{ro}(A) \in \mathbb{N}_\Delta^n \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\zeta_r(\theta_A^+)[\text{diag}(\lambda)] = \begin{cases} \theta_{A+\text{diag}(\lambda-\text{co}(A)),r} & \text{if } \lambda - \text{co}(A) \in \mathbb{N}_\Delta^n \\ 0 & \text{otherwise.} \end{cases}$$

for $\lambda \in \Lambda_\Delta(n, r)$.

For $A, B \in \Theta_\Delta(n, r)$ we write

$$(4.5) \quad \theta_{A,r} \theta_{B,r} = \sum_{C \in \Theta_\Delta(n, r)} \mathfrak{g}_{A,B,C,r} \theta_{C,r}$$

where $\mathfrak{g}_{A,B,C,r} \in \mathcal{Z}$. If $\mathfrak{g}_{A,B,C,r} \neq 0$ then we have $\text{co}(A) = \text{ro}(B)$, $\text{ro}(A) = \text{ro}(C)$ and $\text{co}(B) = \text{co}(C)$.

Lemma 4.4. *Let $A, B \in \Theta_\Delta^+(n)$, $\lambda \in \Lambda_\Delta(n, r - \sigma(A))$ and $\mu \in \Lambda_\Delta(n, r - \sigma(B))$. If $\text{co}(A) + \lambda = \text{ro}(B) + \mu$ then we have*

$$\mathfrak{g}_{A+\text{diag}(\lambda), B+\text{diag}(\mu), C', r} = \begin{cases} \mathfrak{f}_{A,B,C} & \text{if } C' = C + \text{diag}(\lambda + \text{ro}(A - C)) \text{ for some } C \in \Theta_\Delta^+(n), \\ 0 & \text{otherwise.} \end{cases}$$

for $C' \in \Theta_\Delta(n, r)$, where $\mathfrak{f}_{A,B,C}$ is as given in (4.3).

Proof. By Lemma 4.3 we have

$$\begin{aligned} \theta_{A+\text{diag}(\lambda), r} \theta_{B+\text{diag}(\mu), r} &= [\text{diag}(\lambda + \text{ro}(A))] \zeta_r(\theta_A^+) \zeta_r(\theta_B^+) [\text{diag}(\mu + \text{co}(B))] \\ &= \sum_{\substack{C \in \Theta_\Delta^+(n), \mathbf{d}(C) = \mathbf{d}(A) + \mathbf{d}(B) \\ \lambda + \text{ro}(A) - \text{ro}(C) \in \mathbb{N}_\Delta^n}} \mathfrak{f}_{A,B,C} \theta_{C+\text{diag}(\lambda + \text{ro}(A) - \text{ro}(C)), r} [\text{diag}(\mu + \text{co}(B))]. \end{aligned}$$

If $\mathbf{d}(C) = \mathbf{d}(A) + \mathbf{d}(B)$ then we have $\text{ro}(C) - \text{co}(C) = \text{ro}(A + B) - \text{co}(A + B)$ and hence $\text{co}(C) + \lambda + \text{ro}(A) - \text{ro}(C) = \lambda + \text{co}(A + B) - \text{ro}(B) = \mu + \text{co}(B)$. Thus we have

$$\theta_{A+\text{diag}(\lambda), r} \theta_{B+\text{diag}(\mu), r} = \sum_{\substack{C \in \Theta_\Delta^+(n), \mathbf{d}(C) = \mathbf{d}(A) + \mathbf{d}(B) \\ \lambda + \text{ro}(A) - \text{ro}(C) \in \mathbb{N}_\Delta^n}} \mathfrak{f}_{A,B,C} \theta_{C+\text{diag}(\lambda + \text{ro}(A) - \text{ro}(C)), r}.$$

The assertion follows. \square

4.3. For $m \in \mathbb{Z}$ there is a map

$$(4.6) \quad \eta_m : \Theta_\Delta(n) \rightarrow \Theta_\Delta(n)$$

defined by sending $A = (a_{i,j})_{i,j \in \mathbb{Z}}$ to $(a_{i, mn+j})_{i,j \in \mathbb{Z}}$. Note that if $A = j_\Delta(\lambda, d, \mu) \in \Theta_\Delta(n, r)$ then $\eta_m(A) = j_\Delta(\lambda, d\rho^{mr}, \mu) \in \Theta_\Delta(n, r)$.

Lemma 4.5. *Let $A \in \Theta_\Delta(n)$ and $m \in \mathbb{Z}$. If $a_{i,j} = 0$ for $1 \leq i \leq n$ and $j \leq mn$, then $\eta_k(A) \in \Theta_\Delta^+(n)$ for $k \leq m - 1$.*

Proof. Let $B^{(k)} = \eta_k(A)$. If $k \leq m-1$, $1 \leq i \leq n$ and $i \geq j$, then $kn + j \leq (m-1)n + j \leq (m-1)n + i \leq mn$ and hence $b_{i,j}^{(k)} = a_{i,kn+j} = 0$. Thus $B^{(k)} \in \Theta_\Delta^+(n)$ for $k \leq m-1$. \square

Lemma 4.6. *Let $A \in \Theta_\Delta(n, r)$ with $\lambda = \text{ro}(A)$ and $\mu \in \text{co}(A)$. Then we have $\theta_{A,r} \cdot \theta_{\mu,\mu}^{\rho^{mr}} = \theta_{\eta_m(A),r} = \theta_{\lambda,\lambda}^{\rho^{mr}} \cdot \theta_{A,r}$ for $m \in \mathbb{Z}$.*

Proof. Note that $C'_{w_{0,\mu}} = v^{-\ell(w_{0,\mu})} x_\mu$. Since $\rho^r x = x \rho^r$ for $x \in \mathfrak{S}_{\Delta,r}$ we have $\mathfrak{S}_\mu \rho^{mr} \mathfrak{S}_\mu = \mathfrak{S}_\mu \mathfrak{S}_\mu \rho^{mr} = \mathfrak{S}_\mu \rho^{mr}$. It follows that $w_{0,\mu} \rho^{mr}$ is the longest element in $\mathfrak{S}_\mu \rho^{mr} \mathfrak{S}_\mu$. This together with Proposition 3.8 implies that

$$(4.7) \quad \theta_{A,r} \theta_{\mu,\mu}^{\rho^{mr}} (C'_{w_{0,\mu}}) = \theta_{A,r} (C'_{w_{0,\mu} \cdot \rho^{mr}}) = \theta_{A,r} (C'_{w_{0,\mu}} T_\rho^{mr}) = C'_{d^+ \rho^{mr}},$$

where $d \in \mathcal{D}_{\lambda,\mu}^\Delta$ is such that $j_\Delta(\lambda, d, \mu) = A$ and d^+ is the unique longest element in $\mathfrak{S}_\lambda d \mathfrak{S}_\mu$. Furthermore since $\mathfrak{S}_\lambda d \rho^{mr} \mathfrak{S}_\mu = \mathfrak{S}_\lambda d \mathfrak{S}_\mu \rho^{mr}$, we see that $d^+ \rho^{mr}$ is the longest element in $\mathfrak{S}_\lambda d \rho^{mr} \mathfrak{S}_\mu$. It follows from (4.7) that

$$\theta_{\eta_m(A),r} (C'_{w_{0,\mu}}) = \theta_{\lambda,\mu}^{d \rho^{mr}} (C'_{w_{0,\mu}}) = C'_{d^+ \rho^{mr}} = \theta_{A,r} \theta_{\mu,\mu}^{\rho^{mr}} (C'_{w_{0,\mu}}).$$

Thus we have $\theta_{A,r} \cdot \theta_{\mu,\mu}^{\rho^{mr}} = \theta_{\eta_m(A),r}$. This implies that $\theta_{\mu,\lambda}^{d^{-1}} \cdot \theta_{\lambda,\lambda}^{\rho^{-mr}} = \theta_{\mu,\lambda}^{d^{-1} \rho^{-mr}}$. Applying the map τ_r given in (2.2), we get $\theta_{\lambda,\lambda}^{\rho^{mr}} \cdot \theta_{A,r} = \tau_r(\theta_{\mu,\lambda}^{d^{-1}} \cdot \theta_{\lambda,\lambda}^{\rho^{-mr}}) = \tau_r(\theta_{\mu,\lambda}^{d^{-1} \rho^{-mr}}) = \theta_{\eta_m(A),r}$. \square

Assume $N \geq n$. There is a natural injective map

$$\sim : \Theta_\Delta(n) \longrightarrow \Theta_\Delta(N), \quad A = (a_{i,j}) \longmapsto \tilde{A} = (\tilde{a}_{i,j}),$$

where $\tilde{A} = (\tilde{a}_{i,j})$ is defined by

$$\tilde{a}_{k,l+mN} = \begin{cases} a_{k,l+mn}, & \text{if } 1 \leq k, l \leq n; \\ 0, & \text{if either } n < k \leq N \text{ or } n < l \leq N \end{cases}$$

for $m \in \mathbb{Z}$. Note that the map $\sim : \Theta_\Delta(n) \longrightarrow \Theta_\Delta(N)$ induces a map from $\Theta_\Delta^+(n)$ to $\Theta_\Delta^+(N)$. Similarly, there is an injective map

$$\sim : \mathbb{Z}_\Delta^n \longrightarrow \mathbb{Z}_\Delta^N, \quad \lambda \longmapsto \tilde{\lambda},$$

where $\tilde{\lambda}_i = \lambda_i$ for $1 \leq i \leq n$ and $\tilde{\lambda}_i = 0$ for $n+1 \leq i \leq N$.

It is easy to see that there is an injective algebra homomorphism (not sending 1 to 1)

$$\iota_{n,N} : \mathcal{S}_\Delta(n, r) \longrightarrow \mathcal{S}_\Delta(N, r), \quad [A] \longmapsto [\tilde{A}] \text{ for } A \in \Theta_\Delta(n, r)$$

(see [DDF, §4.1]).

Let $\Theta_\Delta(n, r)^{\text{ap}} = \Theta_\Delta(n)^{\text{ap}} \cap \Theta_\Delta(n, r)$.

Lemma 4.7. *Assume $N > n$. Then for $A \in \Theta_\Delta(n, r)$ we have $\tilde{A} \in \Theta_\Delta(N, r)^{\text{ap}}$ and $\iota_{n,N}(\theta_{A,r}) = \theta_{\tilde{A},r}$. In particular we have $\mathfrak{g}_{A,B,C,r} = \mathfrak{g}_{\tilde{A},\tilde{B},\tilde{C},r}$ for $A, B, C \in \Theta_\Delta(n, r)$, where $\mathfrak{g}_{A,B,C,r}$ is as given in (4.5).*

Proof. The first assertion follows from the definition of \tilde{A} . The second assertion follows from Proposition 3.8 and (3.3). \square

Recall the map η_m defined in (4.6). The structure constants for the canonical basis $\mathbf{B}(n, r) = \{\theta_{A,r} \mid A \in \Theta_\Delta(n, r)\}$ of the affine quantum Schur algebra $\mathcal{S}_\Delta(n, r)$ and the canonical basis $\mathbf{B}(N)^{\text{ap}} = \{\theta_A^+ \mid A \in \Theta_\Delta^+(N)^{\text{ap}}\}$ of $\mathbf{U}(\widehat{\mathfrak{sl}}_N)^+$ are related as follows.

Theorem 4.8. *Assume $N \geq n$. Let $A, B, C \in \Theta_\Delta(n, r)$ and $C' \in \Theta_\Delta(N, r)$.*

(1) *We have*

$$\mathbf{g}_{\widetilde{\eta_k(A)}, \widetilde{\eta_k(B)}, \widetilde{C'}, r} = \begin{cases} \mathbf{g}_{A, B, X, r} & \text{if } C' = \eta_{2k}(X) \text{ for some } X \in \Theta_\Delta(n, r) \\ 0 & \text{otherwise} \end{cases}$$

for $k \in \mathbb{Z}$, where $\mathbf{g}_{A, B, X, r}$ is as given in (4.5).

(2) *If $N > n$ and $\text{co}(A) = \text{ro}(B)$, then there exist $k_0 \in \mathbb{Z}$ such that for $k \leq k_0$, $\widetilde{\eta_k(A)}, \widetilde{\eta_k(B)}, \widetilde{\eta_{2k}(C)} \in \Theta_\Delta^+(N) \cap \Theta_\Delta(N, r)^{\text{ap}}$ and $\mathbf{g}_{A, B, C, r} = \mathbf{f}_{\widetilde{\eta_k(A)}, \widetilde{\eta_k(B)}, \widetilde{\eta_{2k}(C)}}$, where $\mathbf{f}_{\widetilde{\eta_k(A)}, \widetilde{\eta_k(B)}, \widetilde{\eta_{2k}(C)}}$ is as given in (4.3).*

Proof. If $\text{co}(A) \neq \text{ro}(B)$ then $\theta_{A,r} \theta_{B,r} = \theta_{\eta_k(A)} \theta_{\eta_k(B)} = 0$ for any $k \in \mathbb{Z}$. Now we assume $\text{co}(A) = \text{ro}(B)$. Let $\lambda = \text{ro}(A)$ and $\nu = \text{co}(B)$. Then by Lemma 4.6 we have

$$(4.8) \quad \theta_{\eta_k(A), r} \theta_{\eta_k(B), r} = \theta_{\lambda, \lambda}^{\rho_{kr}} \theta_{A, r} \theta_{B, r} \theta_{\nu, \nu}^{\rho_{kr}} = \sum_{X \in \Theta_\Delta(n, r)} \mathbf{g}_{A, B, X, r} \theta_{\eta_k(X), r} \theta_{\nu, \nu}^{\rho_{kr}} = \sum_{X \in \Theta_\Delta(n, r)} \mathbf{g}_{A, B, X, r} \theta_{\eta_{2k}(X), r}$$

for $k \in \mathbb{Z}$. Applying $\iota_{n, N}$ to (4.8) gives that

$$\iota_{n, N}(\theta_{\eta_k(A), r}) \iota_{n, N}(\theta_{\eta_k(B), r}) = \sum_{X \in \Theta_\Delta(n, r)} \mathbf{g}_{A, B, X, r} \iota_{n, N}(\theta_{\eta_{2k}(X), r})$$

for $k \in \mathbb{Z}$. Thus by Lemma 4.7 we have

$$(4.9) \quad \theta_{\widetilde{\eta_k(A)}, r} \theta_{\widetilde{\eta_k(B)}, r} = \sum_{X \in \Theta_\Delta(n, r)} \mathbf{g}_{A, B, X, r} \theta_{\widetilde{\eta_{2k}(X)}, r}$$

for $k \in \mathbb{Z}$. The assertion (1) follows. The assertion (2) follows from the assertion (1), Lemma 4.4, Lemma 4.5 and Lemma 4.7. \square

As a corollary to Theorem 4.8, together with Theorem 4.1 we have the following positivity property for $\mathcal{S}_\Delta(n, r)$. This gives an alternate approach to Lusztig's result on positivity property for $\mathcal{S}_\Delta(n, r)$ in [L6, 4.5].

Corollary 4.9. *For $A, B, C \in \Theta_\Delta(n, r)$ we have $\mathbf{g}_{A, B, C, r} \in \mathbb{N}[v, v^{-1}]$.*

4.4. There is an injective map from $\mathbf{B}(n)$ to $\mathbf{B}(N)^{\text{ap}}$ defined by sending θ_A^+ to θ_A^+ for $A \in \Theta_\Delta^+(n)$. The structure constants for the canonical basis $\mathbf{B}(n)$ of $\mathfrak{D}_\Delta(n)_\mathbb{Z}^+$ and the canonical basis $\mathbf{B}(N)^{\text{ap}}$ of $U(\widehat{\mathfrak{sl}}_N)_\mathbb{Z}^+$ are related as follows.

Theorem 4.10. *Assume $N > n$. For $A, B, C \in \Theta_\Delta^+(n)$ we have $f_{A,B,C} = f_{\tilde{A},\tilde{B},\tilde{C}}$, where $f_{A,B,C}$ is as given in (4.3).*

Proof. There exist $\lambda, \mu \in \mathbb{N}_\Delta^n$ such that $\lambda + \text{co}(A) = \mu + \text{ro}(B)$ and $\lambda + \text{ro}(A) - \text{ro}(C) \in \mathbb{N}_\Delta^n$. Let $r = \sigma(\lambda) + \sigma(A)$. Then by Lemma 4.4 and Lemma 4.7 we have $f_{A,B,C} = \mathfrak{g}_{A+\text{diag}(\lambda), B+\text{diag}(\mu), C+\text{diag}(\lambda+\text{ro}(A-C))} = \mathfrak{g}_{\tilde{A}+\text{diag}(\tilde{\lambda}), \tilde{B}+\text{diag}(\tilde{\mu}), \tilde{C}+\text{diag}(\tilde{\lambda}+\text{ro}(\tilde{A}-\tilde{C}))} = f_{\tilde{A},\tilde{B},\tilde{C}}$. \square

The following result is a generalization of Theorem 4.1, which gives the positivity property for $\mathfrak{D}_\Delta(n)_\mathbb{Z}^+$.

Corollary 4.11. *For $A, B, C \in \Theta_\Delta^+(n)$ we have $f_{A,B,C} \in \mathbb{N}[v, v^{-1}]$.*

Proof. The assertion follows from Theorem 4.1 and Theorem 4.10. \square

5. POSITIVITY PROPERTIES FOR $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$

5.1. Recall that $I = \mathbb{Z}/n\mathbb{Z}$ and I is identified with $\{1, 2, \dots, n\}$. There is an algebra grading over $\mathbb{Z}[I]$

$$\mathbf{U}(\widehat{\mathfrak{sl}}_n) = \bigoplus_{\nu \in \mathbb{Z}[I]} \mathbf{U}(\widehat{\mathfrak{sl}}_n)_\nu$$

defined by the condition $\mathbf{U}(\widehat{\mathfrak{sl}}_n)_{\nu'} \mathbf{U}(\widehat{\mathfrak{sl}}_n)_{\nu''} \subseteq \mathbf{U}(\widehat{\mathfrak{sl}}_n)_{\nu'+\nu''}$, $\tilde{K}_i \in \mathbf{U}(\widehat{\mathfrak{sl}}_n)_0$, $u_{E_{i,i+1}^\Delta}^+ \in \mathbf{U}(\widehat{\mathfrak{sl}}_n)_i$, $u_{E_{i+1,i}^\Delta}^- \in \mathbf{U}(\widehat{\mathfrak{sl}}_n)_{-i}$ for all $\nu', \nu'' \in \mathbb{Z}[I]$, $i \in I$.

Let us recall the definition of the modified quantum affine algebra $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$ of $\mathbf{U}(\widehat{\mathfrak{sl}}_n)$. Let X be the quotient of \mathbb{Z}_Δ^n by the subgroup generated by the element $\mathbf{1}$, where $\mathbf{1}_i = 1$ for all i . For $\lambda \in \mathbb{Z}_\Delta^n$ let $\bar{\lambda} \in X$ be the image of λ in X . Let $Y = \{\mu \in \mathbb{Z}_\Delta^n \mid \sum_{1 \leq i \leq n} \mu_i = 0\}$. For $\bar{\lambda} \in X$ and $\mu \in Y$ we set $\mu \cdot \bar{\lambda} = \sum_{1 \leq i \leq n} \lambda_i \mu_i$.

For $i \in I$ let $e_i^\Delta \in \mathbb{N}_\Delta^n$ be the element satisfying $(e_i^\Delta)_j = \delta_{i,j}$ for $j \in I$. There is a natural map $I \rightarrow X$ defined by sending i to $\overline{\alpha_i^\Delta}$, where $\alpha_i^\Delta = e_i^\Delta - e_{i+1}^\Delta$. The imbedding $I \rightarrow X$ induce a homomorphism $\iota : \mathbb{Z}[I] \rightarrow X$.

For $\bar{\lambda}, \bar{\mu} \in X$ we set

$$\bar{\lambda} \mathbf{U}(\widehat{\mathfrak{sl}}_n)_{\bar{\mu}} = \mathbf{U}(\widehat{\mathfrak{sl}}_n) / \left(\sum_{j \in Y} (K^j - v^{j \cdot \bar{\lambda}}) \mathbf{U}(\widehat{\mathfrak{sl}}_n) + \sum_{j \in Y} \mathbf{U}(\widehat{\mathfrak{sl}}_n) (K^j - v^{j \cdot \bar{\mu}}) \right).$$

Let $\pi_{\bar{\lambda}, \bar{\mu}} : \mathbf{U}(\widehat{\mathfrak{sl}}_n) \rightarrow \bar{\lambda} \mathbf{U}(\widehat{\mathfrak{sl}}_n)_{\bar{\mu}}$ be the canonical projection. Let

$$\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n) := \bigoplus_{\bar{\lambda}, \bar{\mu} \in X} \bar{\lambda} \mathbf{U}(\widehat{\mathfrak{sl}}_n)_{\bar{\mu}}.$$

We define the product in $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$ as follows. Let $\lambda', \mu', \lambda'', \mu'' \in X$ and $\nu', \nu'' \in \mathbb{Z}[I]$ with $\lambda' - \mu' = \iota(\nu')$ and $\lambda'' - \mu'' = \iota(\nu'')$. For $t \in \mathbf{U}(\widehat{\mathfrak{sl}}_n)_{\nu'}$, $s \in \mathbf{U}(\widehat{\mathfrak{sl}}_n)_{\nu''}$, define

$$\pi_{\lambda', \mu'}(t) \pi_{\lambda'', \mu''}(s) = \begin{cases} \pi_{\lambda', \mu''}(ts), & \text{if } \mu' = \lambda'' \\ 0 & \text{otherwise.} \end{cases}$$

Then $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$ becomes an associative $\mathbb{Q}(v)$ -algebra structure with respect to the above product.

5.2. Let $\mathfrak{D}'_{\Delta}(n)$ be the subalgebra of $\mathfrak{D}_{\Delta}(n)$ generated by the elements u_A^+ , u_A^- and $\tilde{K}_i^{\pm 1}$ for $A \in \Theta_{\Delta}^+(n)$ and $i \in I$. The algebra $\mathfrak{D}'_{\Delta}(n)$ is a $\mathbb{Z}[I]$ -graded algebra with

$$\deg(u_A^+) = \sum_{1 \leq i \leq n} d_i i, \quad \deg(u_A^-) = - \sum_{1 \leq i \leq n} d_i i \quad \text{and} \quad \deg(\tilde{K}_i^{\pm 1}) = 0$$

for $A \in \Theta_{\Delta}^+(n)$ and $1 \leq i \leq n$, where $(d_i)_{i \in \mathbb{Z}} = \mathbf{d}(A)$.

Let

$$\dot{\mathfrak{D}}'_{\Delta}(n) := \bigoplus_{\bar{\lambda}, \bar{\mu} \in X} \bar{\lambda} \mathfrak{D}'_{\Delta}(n)_{\bar{\mu}},$$

where $\bar{\lambda} \mathfrak{D}'_{\Delta}(n)_{\bar{\mu}} = \mathfrak{D}'_{\Delta}(n) / (\sum_{\mathbf{j} \in Y} (K^{\mathbf{j}} - v^{\mathbf{j} \cdot \bar{\lambda}}) \mathfrak{D}'_{\Delta}(n) + \sum_{\mathbf{j} \in Y} \mathfrak{D}'_{\Delta}(n) (K^{\mathbf{j}} - v^{\mathbf{j} \cdot \bar{\mu}}))$. As in the case of $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$, there is a natural associative $\mathbb{Q}(v)$ -algebra structure on $\dot{\mathfrak{D}}'_{\Delta}(n)$ inherited from that of $\mathfrak{D}'_{\Delta}(n)$. We will naturally regard $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$ as a subalgebra of $\dot{\mathfrak{D}}'_{\Delta}(n)$.

For $\bar{\lambda}, \bar{\mu} \in X$, let $\pi_{\bar{\lambda}, \bar{\mu}} : \mathfrak{D}'_{\Delta}(n) \rightarrow \bar{\lambda} \mathfrak{D}'_{\Delta}(n)_{\bar{\mu}}$ be the canonical projection. The algebra $\dot{\mathfrak{D}}'_{\Delta}(n)$ is naturally a $\mathfrak{D}'_{\Delta}(n)$ -bimodule defined by

$$t' \pi_{\lambda', \lambda''}(s) t'' = \pi_{\lambda' + \iota(\nu'), \lambda'' - \iota(\nu'')}(t' s t'')$$

for $t' \in \mathfrak{D}'_{\Delta}(n)_{\nu'}$, $s \in \mathfrak{D}'_{\Delta}(n)$, $t'' \in \mathfrak{D}'_{\Delta}(n)_{\nu''}$ and $\lambda', \lambda'' \in X$.

For $\bar{\lambda} \in X$ let $1_{\bar{\lambda}} = \pi_{\bar{\lambda}, \bar{\lambda}}(1)$. The map ζ_r defined in Theorem 2.2 induces an surjective algebra homomorphism

$$\dot{\zeta}_r : \dot{\mathfrak{D}}'_{\Delta}(n) \rightarrow \mathcal{S}_{\Delta}(n, r)$$

such that for $A \in \Theta_{\Delta}^+(n)$ and $\bar{\lambda} \in X$, $\dot{\zeta}_r(u_A^{\pm} 1_{\bar{\lambda}}) = \zeta_r(u_A^{\pm})[\text{diag}(\mu)]$, if $\bar{\lambda} = \bar{\mu}$ for some $\mu \in \Lambda_{\Delta}(n, r)$, and $\dot{\zeta}_r(u_A^{\pm} 1_{\bar{\lambda}}) = 0$ otherwise (cf. [F1, 3.6]).

The maps $\dot{\zeta}_r$ induce an algebra homomorphism

$$\dot{\zeta} : \dot{\mathfrak{D}}'_{\Delta}(n) \rightarrow \prod_{r \geq 0} \mathcal{S}_{\Delta}(n, r)$$

such that $\dot{\zeta}(x) = (\dot{\zeta}_r(x))_{r \geq 0}$ for $x \in \dot{\mathfrak{D}}'_{\Delta}(n)$. The following result is a generalization of Lusztig [L7, 3.5].

Theorem 5.1. *The map $\dot{\zeta} : \dot{\mathfrak{D}}'_{\Delta}(n) \rightarrow \prod_{r \geq 0} \mathcal{S}_{\Delta}(n, r)$ is injective.*

Proof. Note that the set $\{1_{\bar{\lambda}}\tilde{u}_A^+\tilde{u}_B^- \mid A, B \in \Theta_\Delta^+(n), \bar{\lambda} \in X\}$ forms a $\mathbb{Q}(v)$ -basis for $\dot{\mathfrak{B}}'_\Delta(n)$. We use reduction to absurdity. Assume $x = \sum_{A \in \Theta_\Delta^\pm(n), \bar{\lambda} \in X} \beta_{A, \bar{\lambda}} 1_{\bar{\lambda}} \tilde{u}_{A^+}^+ \tilde{u}_{(A^-)}^- \neq 0 \in \dot{\mathfrak{B}}'_\Delta(n)$ is such that $\dot{\zeta}(x) = 0$. Then there exist $\mathbf{a} \in X$ such that $1_{\mathbf{a}}x \neq 0$. Since the set

$$\mathcal{T} := \{A \mid A \in \Theta_\Delta^\pm(n), \beta_{A, \mathbf{a}} \neq 0\}$$

is finite we may choose a maximal element B in \mathcal{T} with respect to \preccurlyeq . We choose $\mu \in \mathbb{N}_\Delta^n$ such that $\bar{\mu} = \mathbf{a}$ and $\mu \geq \text{ro}(B)$. Let $r_0 = \sigma(\mu)$. Then we have

$$0 = [\text{diag}(\mu)]\dot{\zeta}_{r_0}(x) = \sum_{A \in \mathcal{T}} \beta_{A, \mathbf{a}} [\text{diag}(\mu)]A^+(\mathbf{0}, r)A^-(\mathbf{0}, r).$$

By [DDF, 3.7.3] we have

$$A^+(\mathbf{0}, r)A^-(\mathbf{0}, r) = A(\mathbf{0}, r) + \sum_{C \in \Theta_\Delta(n, r), C \prec A} \gamma_{A, C}[C]$$

where $\gamma_{A, C} \in \mathbb{Q}(v)$. This implies that

$$\begin{aligned} & \sum_{A \in \mathcal{T}} \beta_{A, \mathbf{a}} [\text{diag}(\mu)]A^+(\mathbf{0}, r)A^-(\mathbf{0}, r) \\ &= \beta_{B, \mathbf{a}} [\text{diag}(\mu)] \left(B(\mathbf{0}, r) + \sum_{\substack{C \in \Theta_\Delta(n, r) \\ C \prec B}} \gamma_{B, C}[C] \right) + \sum_{\substack{A \in \mathcal{T} \\ B \not\prec A}} \beta_{A, \mathbf{a}} [\text{diag}(\mu)] \left(A(\mathbf{0}, r) + \sum_{\substack{C \in \Theta_\Delta(n, r) \\ C \prec A}} \gamma_{A, C}[C] \right) \\ &= \beta_{B, \mathbf{a}} [B + \text{diag}(\mu - \text{ro}(B))] + f \end{aligned}$$

where f is a linear combination of $[C' + \text{diag}(\nu)]$ such that $C' \neq B$, $C' \in \Theta_\Delta^\pm(n)$ and $\nu \in \Theta_\Delta(n, r - \sigma(C'))$. Thus we have $\beta_{B, \mathbf{a}} = 0$. This is a contradiction. \square

5.3. Let $\dot{\mathbf{B}}(n)$ be the canonical basis of $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$ defined in [L5]. Let $\phi_{r+n, r} : \mathcal{S}_\Delta(n, r+n) \rightarrow \mathcal{S}_\Delta(n, r)$ be the algebra homomorphism defined in [L7, 1.11]. According to [L7, 3.4(a)] we have

$$(5.1) \quad \phi_{r+n, r} \circ \dot{\zeta}_{r+n}(x) = \dot{\zeta}_r(x)$$

for all $r \in \mathbb{N}$ and $x \in \dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$. The following result was proved by Schiffmann–Vasserot [SV] (see also Lusztig [L7, 4.1] and McGerty [M, 7.10]).

Theorem 5.2. (1) We have $\dot{\zeta}_r(\dot{\mathbf{B}}(n)) \subseteq \{0\} \cup \{\theta_{A, r} \mid A \in \Theta_\Delta(n, r)\}$.

(2) For $A \in \Theta_\Delta(n, r+n)^{\text{ap}}$ we have

$$\phi_{r+n, r}(\theta_{A, r+n}) = \begin{cases} \theta_{A-E, r} & \text{if } a_{i, i} \geq 1 \text{ for } 1 \leq i \leq n, \\ 0 & \text{otherwise,} \end{cases}$$

where $E = (\delta_{i, j})_{i, j \in \mathbb{Z}} \in \Theta_\Delta(n)$.

For $A \in \Theta_\Delta(n)^{\text{ap}}$ with $A - E \notin \Theta_\Delta(n)$ let $\mathbf{b}_A = (a_r)_{r \geq 0} \in \prod_{r \geq 0} \mathcal{S}_\Delta(n, r)$, where $a_r = \theta_{A+mE, r}$ if $r = \sigma(A) + mn$ for some $m \geq 0$, and $a_r = 0$ otherwise.

Lemma 5.3. *We have $\dot{\zeta}(\dot{\mathbf{B}}(n)) = \{\mathbf{b}_A \mid A \in \Theta_\Delta(n)^{\text{ap}}, A - E \notin \Theta_\Delta(n)\}$.*

Proof. Let $b \in \dot{\mathbf{B}}(n)$. By Theorem 5.1 we have $\dot{\zeta}(b) \neq 0$. Let $r_0 = \min\{r \in \mathbb{N} \mid \dot{\zeta}_r(b) \neq 0\}$. Then by Theorem 5.2(1) and [L6, 8.2] we have $\dot{\zeta}_{r_0}(b) = \theta_{A,r_0}$ for some $A \in \Theta_\Delta(n, r_0)^{\text{ap}}$. From (5.1) we see that $\phi_{r_0, r_0-n}(\theta_{A,r_0}) = \phi_{r_0, r_0-n} \circ \dot{\zeta}_{r_0}(b) = \dot{\zeta}_{r_0-n}(b) = 0$. Thus by Theorem 5.2(2) we have $A - E \notin \Theta_\Delta(n)$. By the proof of [L7, 4.3], we know that if $\dot{\zeta}_r(b) \neq 0$ for some $r > r_0$, then $r \equiv r_0 \pmod n$. Furthermore, if $m > 0$ then by (5.1) we have

$$\theta_{A,r_0} = \dot{\zeta}_{r_0}(b) = \phi_{r_0+n, r_0} \circ \phi_{r_0+2n, r_0+n} \circ \cdots \circ \phi_{r_0+mn, r_0+(m-1)n} \circ \dot{\zeta}_{r_0+mn}(b).$$

This together with Theorem 5.2 implies that $\dot{\zeta}_{r_0+mn}(b) = \theta_{A+mE, r_0+mn}$. Thus we have $\dot{\zeta}(b) = \mathbf{b}_A$.

On the other hand, if $A' \in \Theta_\Delta(n)^{\text{ap}}$ with $A' - E \notin \Theta_\Delta(n)$, by [L6, 8.2] we conclude that there exists $b' \in \dot{\mathbf{B}}(n)$ such that $\dot{\zeta}_{r'_0}(b') = \theta_{A', r'_0}$, where $r'_0 = \sigma(A')$. By the proof above we conclude that $\dot{\zeta}(b') = \mathbf{b}_{A'}$. The assertion follows. \square

By Theorem 5.1 and Lemma 5.3 we conclude that for each $A \in \Theta_\Delta(n)^{\text{ap}}$ with $A - E \notin \Theta_\Delta(n)$, there exists a unique $\mathbf{c}_A \in \dot{\mathbf{B}}(n)$ such that $\dot{\zeta}(\mathbf{c}_A) = \mathbf{b}_A$. Furthermore we have

$$\dot{\mathbf{B}}(n) = \{\mathbf{c}_A \mid A \in \Theta_\Delta(n)^{\text{ap}}, A - E \notin \Theta_\Delta(n)\}.$$

Thus $\dot{\mathbf{B}}(n)$ is indexed by the set $\{A \in \Theta_\Delta(n)^{\text{ap}} \mid A - E \notin \Theta_\Delta(n)\}$. For $A, B \in \Theta_\Delta(n)^{\text{ap}}$ with $A - E, B - E \notin \Theta_\Delta(n)$ we write

$$(5.2) \quad \mathbf{c}_A \mathbf{c}_B = \sum_{\substack{C \in \Theta_\Delta(n)^{\text{ap}} \\ C - E \notin \Theta_\Delta(n)}} \mathbf{h}_{A,B,C} \mathbf{c}_C,$$

where $\mathbf{h}_{A,B,C} \in \mathcal{Z}$.

Recall the map η_m defined in (4.6). The structure constants for the canonical basis $\dot{\mathbf{B}}(n)$ of $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$ and the structure constants for the canonical basis $\mathbf{B}(N)^{\text{ap}} = \{\theta_A^+ \mid A \in \Theta_\Delta^+(N)^{\text{ap}}\}$ of $\mathbf{U}(\widehat{\mathfrak{sl}}_N)^+$ are related in the following way.

Theorem 5.4. *Assume $N > n$. Let $A, B \in \Theta_\Delta(n)^{\text{ap}}$ with $A - E, B - E \notin \Theta_\Delta(n)$. If $C \in \Theta_\Delta(n)^{\text{ap}}$ with $C - E \notin \Theta_\Delta(n)$ is such that $\mathbf{h}_{A,B,C} \neq 0$, then there exist $m_1, m_2, m_C \in \mathbb{N}$ and $k_0 \in \mathbb{Z}$ such that $\sigma(A) + nm_1 = \sigma(B) + nm_2 = \sigma(C) + nm_C$, $\widetilde{A}_k, \widetilde{B}_k, \widetilde{C}_k \in \Theta_\Delta^+(N)^{\text{ap}}$ and*

$$\mathbf{h}_{A,B,C} = \mathbf{f}_{\widetilde{A}_k, \widetilde{B}_k, \widetilde{C}_k}$$

for $k \leq k_0$, where $A_k = \eta_k(A + m_1 E)$, $B_k = \eta_k(B + m_2 E)$, $C_k = \eta_{2k}(C + m_C E)$ and $\mathbf{f}_{\widetilde{A}_k, \widetilde{B}_k, \widetilde{C}_k}$ is as given in (4.3).

Proof. By (5.2) we have

$$(5.3) \quad \mathbf{b}_A \mathbf{b}_B = \sum_{\substack{C \in \Theta_\Delta(n)^{\text{ap}} \\ C - E \notin \Theta_\Delta(n)}} \mathbf{h}_{A,B,C} \mathbf{b}_C,$$

where $\mathbf{h}_{A,B,C} \in \mathcal{Z}$. If $\sigma(A) \not\equiv \sigma(B) \pmod n$ then by definition we have $\mathbf{b}_A \mathbf{b}_B = 0$. Now we assume $\sigma(A) \equiv \sigma(B) \pmod n$. Let $\mathcal{X} = \{C \in \Theta_\Delta(n)^{\text{ap}} \mid C - E \notin \Theta_\Delta(n), \mathbf{h}_{A,B,C} \neq 0\}$. We choose $r_0 \in \mathbb{N}$ such that $r_0 \equiv \sigma(A) \pmod n$, $r_0 \geq \sigma(A)$, $r_0 \geq \sigma(B)$ and $r_0 \geq \sigma(C)$ for $C \in \mathcal{X}$. Note that $\sigma(C) \equiv \sigma(A) \pmod n$ for $C \in \mathcal{X}$. Assume $r_0 = \sigma(A) + nm_1 = \sigma(B) + nm_2 = \sigma(C) + nm_C$ for $C \in \mathcal{X}$. Then by (5.3) we have

$$\theta_{A+m_1E, r_0} \theta_{B+m_2E, r_0} = \sum_{C \in \mathcal{X}} \mathbf{h}_{A,B,C} \theta_{C+m_CE, r_0}.$$

This implies that $\mathbf{h}_{A,B,C} = \mathbf{g}_{A+m_1E, B+m_2E, C+m_CE, r_0}$. Now the assertion follows from Theorem 4.8. \square

The following theorem gives the positivity property for $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$.

Theorem 5.5. *For $b, b' \in \dot{\mathbf{B}}(n)$ we have $bb' \in \sum_{b'' \in \dot{\mathbf{B}}(n)} \mathbb{N}[v, v^{-1}]b''$.*

Proof. The assertion follows from Theorem 4.1 and Theorem 5.4. \square

6. A WEAK POSITIVITY PROPERTY FOR $\dot{\mathfrak{D}}_\Delta(n)$

For $\lambda, \mu \in \mathbb{Z}_\Delta^n$ we set ${}_\lambda \dot{\mathfrak{D}}_\Delta(n)_\mu = \dot{\mathfrak{D}}_\Delta(n) / {}_\lambda I_\mu$, where

$${}_\lambda I_\mu = \left(\sum_{\mathbf{j} \in \mathbb{Z}_\Delta^n} (K^{\mathbf{j}} - v^{\lambda \cdot \mathbf{j}}) \dot{\mathfrak{D}}_\Delta(n) + \sum_{\mathbf{j} \in \mathbb{Z}_\Delta^n} \dot{\mathfrak{D}}_\Delta(n) (K^{\mathbf{j}} - v^{\mu \cdot \mathbf{j}}) \right).$$

Let $\dot{\mathfrak{D}}_\Delta(n) := \bigoplus_{\lambda, \mu \in \mathbb{Z}_\Delta^n} {}_\lambda \dot{\mathfrak{D}}_\Delta(n)_\mu$. As in the case of $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$, there is a natural associative $\mathbb{Q}(v)$ -algebra structure on $\dot{\mathfrak{D}}_\Delta(n)$ inherited from that of $\dot{\mathfrak{D}}_\Delta(n)$ (see [F1]). The algebra $\dot{\mathfrak{D}}_\Delta(n)$ is the modified form of $\dot{\mathfrak{D}}_\Delta(n)$. Let $\{\theta_A \mid A \in \widetilde{\Theta}_\Delta(n)\}$ be the canonical basis of $\dot{\mathfrak{D}}_\Delta(n)$ defined in [DF3], where $\widetilde{\Theta}_\Delta(n)$ is given in §1.

Proposition 6.1 ([DF3, 7.7]). *There is a surjective algebra homomorphism $\dot{\xi}_r : \dot{\mathfrak{D}}_\Delta(n) \rightarrow \mathcal{S}_\Delta(n, r)$ such that*

$$\dot{\xi}_r(\theta_A) = \begin{cases} \theta_{A,r}, & \text{if } A \in \Theta_\Delta(n, r); \\ 0, & \text{otherwise.} \end{cases}$$

The maps $\dot{\xi}_r$ induce an algebra homomorphism

$$\dot{\xi} : \dot{\mathfrak{D}}_\Delta(n) \rightarrow \prod_{r \geq 0} \mathcal{S}_\Delta(n, r)$$

such that $\dot{\xi}(x) = (\dot{\xi}_r(x))_{r \geq 0}$ for $x \in \dot{\mathfrak{D}}_\Delta(n)$. Contrast to Theorem 5.1, the map $\dot{\xi}$ is not injective. For $A \in \widetilde{\Theta}_\Delta(n)$ let $\overline{\theta_A} = \theta_A + \ker(\dot{\xi}) \in \dot{\mathfrak{D}}_\Delta(n) / \ker(\dot{\xi})$.

Lemma 6.2. *We have $\overline{\theta_A} = 0$ for $A \notin \Theta_\Delta(n)$ and the set $\{\overline{\theta_A} \mid A \in \Theta_\Delta(n)\}$ forms a $\mathbb{Q}(v)$ -basis for $\dot{\mathfrak{D}}_\Delta(n) / \ker \dot{\xi}$.*

Proof. From Proposition 6.1 we see that $\ker \dot{\xi} = \text{span}_{\mathbb{Q}(v)}\{\theta_A \mid A \in \tilde{\Theta}_\Delta(n), A \notin \Theta_\Delta(n)\}$. The assertion follows. \square

The following result gives a weak version of the positivity property for $\dot{\mathfrak{D}}_\Delta(n)$.

Theorem 6.3. *For $A, B \in \Theta_\Delta(n)$ we have $\overline{\theta_A} \cdot \overline{\theta_B} \in \sum_{C \in \Theta_\Delta(n)} \mathbb{N}[v, v^{-1}] \overline{\theta_C}$.*

Proof. The assertion follows from Corollary 4.9, Proposition 6.1 and Lemma 6.2. \square

REFERENCES

- [BLM] A. A. Beilinson, G. Lusztig and R. MacPherson, *A geometric setting for the quantum deformation of GL_n* , Duke Math. J. **61** (1990), 655–677.
- [C] C. W. Curtis, *On Lusztig's isomorphism theorem for Hecke algebras*, J. Algebra **92** (1985), 348–365.
- [DDPW] B. Deng, J. Du, B. Parshall and J.-p. Wang, *Finite dimensional algebras and quantum groups*, Mathematical Surveys and Monographs, Vol. **150**, Amer. Math. Soc. (2008).
- [DDF] B. Deng, J. Du and Q. Fu, *A double Hall algebra approach to affine quantum Schur–Weyl theory*, London Mathematical Society Lecture Note Series, **401**, Cambridge University Press, 2012.
- [D] J. Du, *Kazhdan-Lusztig bases and isomorphism theorems for q -Schur algebras*, Contemp. Math. **139** (1992), 121–140.
- [DF1] J. Du and Q. Fu, *A modified BLM approach to quantum affine \mathfrak{gl}_n* , Math. Z. **266** (2010), 747–781.
- [DF2] J. Du and Q. Fu, *Quantum affine \mathfrak{gl}_n via Hecke algebras*, preprint, arXiv:1311.1868.
- [DF3] J. Du and Q. Fu, *The Integral quantum loop algebra of \mathfrak{gl}_n* , preprint, arXiv:1404.5679.
- [F1] Q. Fu, *Integral affine Schur–Weyl reciprocity*, Adv. Math. **243** (2013), 1–21.
- [F2] Q. Fu, *Canonical bases for modified quantum \mathfrak{gl}_n and q -Schur algebras*, J. Algebra, **406** (2014), 308–320.
- [GV] V. Ginzburg and E. Vasserot, *Langlands reciprocity for affine quantum groups of type A_n* , Internat. Math. Res. Notices 1993, 67–85.
- [G1] R. M. Green, *Positivity properties for q -Schur algebras*, Math. Proc. Cambridge Philos. Soc. **122** (1997), 401–414.
- [G2] R. M. Green, *The affine q -Schur algebra*, J. Algebra **215** (1999), 379–411.
- [K] M. Kashiwara, *On crystal bases of the q -analogue of universal enveloping algebras*, Duke Math. J. **63** (1991), 465–516.
- [KL] D. Kazhdan and G. Lusztig, *Representations of Coxeter groups and Hecke Algebras*, Invent. Math. **53** (1979), 165–184.
- [L1] G. Lusztig, *Canonical bases arising from quantized enveloping algebras*, J. Amer. Math. Soc. **3** (1990), 447–498.
- [L2] G. Lusztig, *Quivers, perverse sheaves, and quantized enveloping algebras*, J. Amer. Math. Soc. **4** (1991), 365–421.
- [L3] G. Lusztig, *Affine quivers and canonical bases*, Inst. Hautes Études Sci. Publ. Math. **76** (1992), 111–163.
- [L4] G. Lusztig, *Canonical bases in tensor products*, Proc. Nat. Acad. Sci. U.S.A. **89** (1992), 8177–8179.
- [L5] G. Lusztig, *Introduction to quantum groups*, Progress in Math. **110**, Birkhäuser, 1993.
- [L6] G. Lusztig, *Aperiodicity in quantum affine \mathfrak{gl}_n* , Asian J. Math. **3** (1999), 147–177.
- [L7] G. Lusztig, *Transfer maps for quantum affine \mathfrak{sl}_n* , in: Representations and quantizations (Shanghai, 1998), China High. Educ. Press, Beijing, 2000, 341–356.

- [M] K. McGerty, *On the geometric realization of the inner product and canonical basis for quantum affine \mathfrak{sl}_n* , Algebra Number Theory **6** (2012), 1097–1131.
- [R] C. M. Ringel, *The composition algebra of a cyclic quiver*, Proc. London Math. Soc. **66** (1993), 507–537.
- [SV] O. Schiffmann and E. Vasserot, *Geometric construction of the global base of the quantum modified algebra of $\widehat{\mathfrak{gl}}_n$* , Transform. Groups **5** (2000), 351–360.
- [VV] M. Varagnolo and E. Vasserot, *On the decomposition matrices of the quantized Schur algebra*, Duke Math. J. **100** (1999), 267–297.
- [X] J. Xiao, *Drinfeld double and Ringel-Green theory of Hall algebras*, J. Algebra **190** (1997), 100–144.

DEPARTMENT OF MATHEMATICS, TONGJI UNIVERSITY, SHANGHAI, 200092, CHINA.

E-mail address: q.fu@hotmail.com, q.fu@tongji.edu.cn

DEPARTMENT OF MATHEMATICS, TONGJI UNIVERSITY, SHANGHAI, 200092, CHINA.

E-mail address: shoji@tongji.edu.cn, shoji@math.nagoya-u.ac.jp